EXISTENCE AND MULTIPLICITY RESULTS FOR NONLOCAL BIHARMONIC AND POLYHARMONIC EQUATIONS

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DOCTOR OF PHILOSOPHY

By

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To My Family

Declaration

I hereby declare that the work reported in the Ph.D. thesis entitled **"Existence And Multiplicity Results For Nonlocal Biharmonic and Polyharmonic Equations"** submitted at **Bennett University, Greater Noida, India**, is an authentic record of my work carried out under the supervision of **Dr. Sarika Goyal**. I have not submitted this work elsewhere for any other degree or diploma. I am fully responsible for the contents of my Ph.D. thesis.

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Supervisor's Certificate

This is to certify that the Ph.D. thesis entitled **"EXISTENCE AND MULTIPLICITY RESULTS FOR NONLOCAL BIHARMONIC AND POLYHARMONIC EQUATIONS"** submitted by **Anu Rani** to **Bennett University, Greater Noida, India** for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Greater Noida *Anu Rani*

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Abstract

The main essence of the thesis is to study the existence and multiplicity results of elliptic equations and elliptic equations involving critical Choquard nonlinear term with sign-changing weight functions. These type of equations appear in the modeling of several real life problems, quantum mechanics, differential geometry, minimal surfaces, diffusion etc. In particular, we deal with the existence and multiplicity results of biharmonic equations, biharmonic system, p-biharmonic equations, and polyharmonic equations with critical Choquard nonlinearity with sign-changing weight functions. Critical problems are difficult to study due to the lack of compactness. To recover the compactness, one needs to study the critical level with the help of minimizers.

We begin with brief literature survey, preliminary results and structure of the thesis in which problems are introduced with brief explanation.

We explore the existence and multiplicity results of the following biharmonic critical Choquard equation

$$(E_\lambda)\begin{cases} \Delta^2 u = \lambda f(x)|u|^{q-2}u + g(x)\left(\displaystyle\int_\Omega \frac{g(y)|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dy\right)|u|^{2^*_\alpha-2}u & \text{in } \Omega, \\ u, \nabla u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 5$, $1 < q < 2$, $0 < \alpha < N$, $2^*_\alpha = \frac{2N-\alpha}{N-4}$ is the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality and $\lambda > 0$ is a parameter. The functions $f, g : \overline{\Omega} \to \mathbb{R}$ are

continuous sign-changing weight functions. We first investigate the minimizers for $S_{H,L}$(defined later) in the case of biharmonic Choquard equation. Then using the Nehari manifold and fibering map analysis, we establish the existence of two non-trivial solutions to the problem (E_λ) with respect to parameter λ.

Then we extend the problem (E_λ) to the biharmonic system involving critical Choquard nonlinearities with sign-changing weight function and prove the existence and multiplicity results of the nontrivial solutions with respect to parameters.

Later, we study the p-biharmonic equation involving Choquard nonlinearity with sign-changing weight functions. Using the Nehari manifold and fibering map analysis, we show the multiplicity results in subcritical case. To deal with the critical case, we first prove the concentration-compactness principle for Choquard equation involving p-biharmonic operator, which is new even in the case of p-Laplace operator. Using this principle, we show the existence results for p-biharmonic equation involving critical Choquard nonlinearity with sign-changing weight functions.

Further, we concern with the existence of multiple solutions of the polyharmonic system involving concave-convex nonlinearities with critical exponent and sign-changing weight functions. We prove that the polyharmonic system admits at least two non-trivial solutions with respect to parameters.

At the end of the thesis, we give the summary of the contributions and the conclusions of the work. Moreover, some future direction of work is also explored.

Contents

List of Symbols

Symbol	Meaning
$\lvert A\rvert$	Lebesgue measure of a set $A \subset \mathbb{R}^N$.
$\mathbf{B}(x,r)$	Ball of radius r centered at x in $\mathbb{R}^N$.
$C_c^\infty(\Omega)$	Set of infinitely differentiable functions with compact support in Ω.
$D^\alpha u$	α^{th} weak derivative of u.
$\lVert f\rVert_p$ or $\lVert f\rVert_{L^p}$	Norm of f in $L^p(\Omega)$.
$W^{m,p}(\Omega)$	Sobolev space of order m and exponent p.
$\lVert u\rVert_{m,p}$	$\sum_{\lvert\alpha\rvert\le m} \lVert D^\alpha u\rVert$, norm of u in $W^{m,p}(\Omega)$.
$W_0^{m,p}(\Omega)$	Closure of $C_c^\infty(\Omega)$ in $W^{m,p}(\Omega)$.
$\lVert u\rVert_{m,p,0}$	$\left(\int_\Omega \lvert D^m u\rvert^p dx\right)^{\frac{1}{p}}$, norm of u in $W_0^{m,p}(\Omega)$.
Δ^2	Biharmonic operator.
Δ^m	Polyharmonic operator/m-harmonic operator.
Δ_p^2	p-biharmonic operator.
$\lVert u\rVert_\infty$	Norm of u in L^∞ space.
$\lVert u\rVert$	Norm of u in the Sobolev space X.
X^{-1}	Dual space of X.

1

Introduction

In previous years, the study of a huge class of partial differential equations has attracted many researchers. Such equations appear in a variety of contexts in Geometry, Mechanics, Physics, and real life problems. The theory has gained a lot of attention in the last few years and its application is scattered throughout thousands of research articles. Elliptic partial differential equations manifest themselves even in the modeling of natural phenomena. So, it is necessary to investigate the existence, uniqueness, and multiplicity of solutions to elliptic equations.

A lot of attention has been specified to the study of biharmonic, p-biharmonic and polyharmonic equations both from concrete applications and from pure mathematical point of view. These models naturally arise in many applications, such as micro-electro-mechanical systems, nonlinear surface diffusion on solids, phase field models of multi-phase systems, thin film theory, interface dynamics, flow in Hele-Shaw cells, and the deformation of a nonlinear elastic beam (see [29, 55]).

Starting with the pioneering work of Ambrosetti et al. [5] on Laplace equations involving convex-concave type nonlinearities, an enormous amount of work has been examined by many authors in this direction (see [6, 16, 19, 42, 45]). Also, a great deal of interest has been shown in case of semilinear equation with positive nonlinearity. But in the last decade, authors are interested to investigate the problems with sign-changing nonlinearity. Nehari manifold technique and fibering map analysis (see [1, 3, 11, 15, 41, 62, 66, 71]) are suitable to deal with the sign-changing problems.

Brézis-Nirenberg type equation has been extensively studied by many authors (see [5, 10, 13, 18, 26, 28, 47]). Edmunds et al. [25] studied the biharmonic equation with critical exponent and showed the existence of a nontrivial solution. Further, Bernis et al. [8] examined the critical growth biharmonic equation with Dirichlet and Navier boundary conditions and showed the infinitely many solutions. Moreover, they proved the existence of at least two positive solutions in the critical case. Afterwards, an ample amount of research has been done in this direction, see [20, 21, 34, 52, 58] and references therein.

Over the past decade, a great mathematical efforts has been devoted to the investigation of biharmonic and p-biharmonic equations (see [2, 9, 23, 39] etc). $\Delta_p^2 u := \Delta\left(|\Delta u|^{p-2}\Delta u\right)$ is known as p-biharmonic operator. For $p = 2$, it is called biharmonc operator (Δ^2), which appears in Navier-Stokes equations as being a viscosity coefficient. Fore more literature on p-biharmonic equation with polynomial type subcritical growth or critical growth (see [17, 46]).

A little while back, an immense attention has been paid in the investigation of Choquard type equations, which arises from the work of S. Pekar in 1976 [57] and P. Choquard. They applied elliptic equations with Hardy-Littlewood-Sobolev type nonlinearity to narrate the model of an electron trapped in its hole in the Hartree-Fock theory of one-component plasma and the quantum theory of a polaron at rest respectively.

Consider the following equation involving Choquard nonlinearity

$$-\Delta u + u = (I_\alpha * F(u))\, F'(u) \ \text{ in } \ \mathbb{R}^N, \tag{1.0.1}$$

where $F \in C^1(\mathbb{R}, \mathbb{R})$, and I_α is known as Riesz potential with $I_\alpha(x) = \frac{B_\alpha}{|x|^{N-\alpha}}$ where $B_\alpha = \frac{\Gamma \frac{N-\alpha}{2}}{\Gamma \frac{\alpha}{2} \pi^{\frac{N}{2}} 2^\alpha}$ and $\alpha \in (0, N)$. For $F(u) = |u|^p$, $\alpha = 2$, $N = 3$ and $p = 2$, problem (1.0.1) has been studied by Choquard-Pekar [48, 57]. For more literature on Choquard equations, we refer to [4, 7, 22, 30, 31, 33, 60, 67] and references therein.

Elliptic system involving nonlinear Choquard term has gained more attention of researchers recently. In [72], You and Zhao proved that the system of nonlinear Choquard equation with Laplace operator has a positive ground state solution using variational methods. We also refer [68, 70] to readers for more information about nonlinear Choquard system. Moreover, for the study of elliptic system involving fractional Laplacian with Choquard nonlinearity, we refer [38, 43, 44]. With this motivation, we are focused to deal the problems with Choquard type and sign-changing nonlinearities having non-compact growth. These problems arise in surface of revolution of minimal area.

The core aspiration of the thesis is to study the existence and multiplicity results of elliptic equations for biharmonic operator and p-biharmonic operator involving critical Choquard nonlinear term with sign-changing weight functions. We begin our work with the investigation of critical Choquard problem involving biharmonic operator and establish the results including the minimizers for $S_{H,L}$ (defined later) not available in the literature so far. Also, we explore this problem to the biharmoic system with critical Hartree type nonlinearity. Further, we study the p-biharmonic equation combined with nonlocal term and provide the concentration-compactness results in the critical case. Moreover, we also study polyharmonic system with critical and sign-changing nonlinearity.

In the next section, we introduce function spaces and Hardy-Littlewood-Sobolev inequality that are used to study all the problems throughout the thesis.

1.1 Function spaces and Hardy-Littlewood-Sobolev inequality

We introduce function spaces and Hardy-Littlewood-Sobolev inequality that are used to study all the problems throughout the thesis.
Consider the model problem

$$(S_\lambda)\begin{cases} (-\Delta)^m u = f_\lambda(x,u) \text{ in } \Omega, \\ D^k u = 0 \text{ for } |k| \le m-1 \text{ on } \partial\Omega, \end{cases}$$

where $(-\Delta)^m$ denotes the polyharmonic operator, $m \in \mathbb{N}$, $f_\lambda(x,u)$ is a Carathéodory function and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. The polyharmonic operator/m-harmonic operator is defined as

$$(-\Delta)^m u := \begin{cases} \Delta^j(\Delta^j u) & \text{if } m = 2j, \quad j = 1,2,\cdots \\ -\nabla.(\Delta^{j-1}\nabla\Delta^{j-1}u) & \text{if } m = 2j-1, \quad j = 1,2,\cdots. \end{cases}$$

- For $m = 1$, $-\Delta$ is a Laplace operator.
- For $m = 2$, Δ^2 is called biharmonic operator.

Now, for $1 \le p \le \infty$, we first introduce the Sobolev space

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), \ \forall\, 0 < |\alpha| \le m\},$$

where $D^\alpha u$ denotes the αth weak derivative. $W^{m,p}(\Omega)$ is a Banach space with respect to norm $\|u\|_{m,p} = \sum_{|\alpha|\le m} \|D^\alpha u\|$.

Then, we define $W_0^{m,p}(\Omega) := \overline{C_c^\infty(\Omega)}$ in $W^{m,p}(\Omega)$ is a Banach space with respect to the norm $\|u\|_{m,p,0} = \left(\int_\Omega |D^m u|^p dx\right)^{\frac{1}{p}}$.

For $p = 2$, we denote $W^{m,p}(\Omega) := H^m(\Omega)$ and $W_0^{m,p}(\Omega) := H_0^m(\Omega)$, where

$$H_0^m(\Omega) := \{u \in H^m(\Omega) : D^\alpha u = 0, \text{on } \partial\Omega \ \ \forall\, 0 \le |\alpha| < m\}.$$

$H_0^m(\Omega)$ is a Hilbert space under the following norm

$$\|D^m u\|^2 = \begin{cases} \|(-\Delta)^{\frac{m}{2}} u\|^2 & \text{if } m = 2j, j = 1, 2, \cdots, \\ \|\nabla(-\Delta)^{\frac{m-1}{2}} u\|^2 & \text{if } m = 2j-1, j = 1, 2, \cdots. \end{cases}$$

The space

$$D^{2,2}(\mathbb{R}^N) := \left\{u \in L^{2^*}(\mathbb{R}^N) : D^\alpha u \in L^2(\mathbb{R}^N),\ \forall\, 0 < |\alpha| \le 2\right\},$$

where $2^* = \frac{2N}{N-4}$, with norm $\|u\|^2 = \int_{\mathbb{R}^N} |\Delta u|^2 dx$, is a Hilbert space.

Further, we introduce the Sobolev embeddings which play a vital role in the study of all problems all over this thesis. So, let $\Omega \subset \mathbb{R}^N$ be bounded and open with $\partial\Omega \in C^1$.

- (i) If $2m < N$, then the embedding $H_0^m(\Omega) \hookrightarrow L^q(\Omega)$ continuous for $1 \le q \le \frac{2N}{N-2m}$ and compact for $1 \le q < \frac{2N}{N-2m}$. The exponent $2_m^* = \frac{2N}{N-2m}$ is known as the critical exponent.
- (ii) If $2m = N$, then the embedding $H_0^m(\Omega) \hookrightarrow L^r(\Omega)$ is compact for $1 \le r < \infty$.
- (iii) If $2m > N$, then the embedding $H_0^m(\Omega) \hookrightarrow C^{\gamma,\alpha}(\overline{\Omega})$ compact if $0 \le \alpha < m - \gamma - \frac{N}{2}$.

Now, we state the well known Hardy-Littlewood-Sobolev inequality which is the central tool to deal with elliptic equation involving Choquard type nonlinearity.

Proposition 1.1.1. *(Hardy-Littlewood-Sobolev inequality, [49]) Let t, $r > 1$ and $0 < \mu < N$ with $1/t + \mu/N + 1/r = 2$, $g \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. Then there exists a sharp constant $C(t, N, \mu, r)$, independent of g, h such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)h(y)}{|x-y|^\mu} dxdy \le C(t, N, \mu, r)\|g\|_{L^t(\mathbb{R}^N)}\|h\|_{L^r(\mathbb{R}^N)}. \tag{1.1.2}$$

If $t = r = \frac{2N}{2N-\mu}$ *then*

$$C(t,N,\mu,r) = C(N,\mu) = \pi^{\frac{\mu}{2}} \frac{\Gamma\left(\frac{N}{2}-\frac{\mu}{2}\right)}{\Gamma\left(N-\frac{\mu}{2}\right)} \left\{ \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)} \right\}^{-1+\frac{\mu}{N}}.$$

In this case there is equality in (1.1.2) *if and only if* $g \equiv Ch$ *and*

$$h(x) = A(b^2 + |x-a|^2)^{-\frac{(2N-\mu)}{2}},$$

for some $A \in \mathbb{C}$, $0 \neq b \in \mathbb{R}$ *and* $a \in \mathbb{R}^N$.

From the Hardy-Littlewood-Sobolev inequality (1.1.2), the integral $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^s|u(y)|^s}{|x-y|^\mu} dxdy$ is well defined if $|u|^s \in L^t(\mathbb{R}^N)$ for $t > 1$ such that $\frac{2}{t} + \frac{\mu}{N} = 2$. Thus for $u \in H^1(\mathbb{R}^N)$, using Sobolev embedding theorems, we obtain

$$\frac{2N-\mu}{N} \leq s \leq \frac{2N-\mu}{N-2} := 2^*_\mu,$$

where $\frac{2N-\mu}{N}$ and $2^*_\mu = \frac{2N-\mu}{N-2}$ are known as lower and upper critical exponent respectively in the sense of Hardy-Littlewood-Sobolev inequality.

1.2 Critical exponent problems

Elliptic partial differential equations having critical exponents have been an interested topic for a long time. We first deliberate some elliptic problems in which the nonlinearity includes critical growth. Such equations are one of the finest cases of loss of compactness and studied by many researchers in last several years. Starting with the work of Brézis-Nirenberg [13] for the following Laplace equation with critical growth

$$(S_\lambda) \begin{cases} -\Delta u = u^{2^*-1} + \lambda u \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega, \end{cases}$$

where Δ is a Laplace operator, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 3$, and $2^* = \frac{2N}{N-2}$ is known as critical Sobolev exponent. The main hurdle was lack of compactness in the embeddings of $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ and due to this Palais-Smale condition is not satisfied. To tackle this difficulty, they studied the critical level below which the Palais-Smale condition is satisfied. Thus, for the existence of solution, some special type of functions $U(x) = (N(N-2))^{\frac{N-2}{4}}(1+|x|^2)^{-\frac{N-2}{2}}$ are studied. These functions were introduced by G. Talenti [65] and known as Talenti type functions. Further, the main results obtained in [13], are the following.

Theorem 1.2.1. *Suppose $\lambda_1 > 0$ is first eigenvalue of $-\Delta$ in Ω.*

(i) For $N \geq 4$, the problem (S_λ) has positive solution iff $\lambda \in (0, \lambda_1)$. If Ω is star shaped, then the problem (S_λ) has no solution for $\lambda \leq 0$.

(ii) For $N = 3$ and $\Omega = B(0,1)$, then the problem (S_λ) has a positive solution iff $\lambda \in \left(\frac{\lambda_1}{4}, \lambda_1\right)$.

Afterwards, D. E. Edmunds [25] investigated the problem (S_λ) for biharmonic operator, then problem (S_λ) can be regarded as biharmonic problem with critical exponent. In this case, let S be the best Sobolev constant for the embedding of $H_0^2(\Omega)$ in $L^{2^*}(\Omega)$ $(2^* = \frac{2N}{N-4})$ defined by

$$S := \inf_{u \in H_0^2(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta u|^2 dx}{\left(\int_\Omega |u(x)|^{2^*} dx\right)^{\frac{2}{2^*}}}.$$

For the minimizers of S, suppose $U(x) = [N(N+2)(N-2)(N-4)]^{\frac{N-4}{8}}(1+|x|^2)^{-\frac{N-4}{2}}$, then all the minimizers of S are obtained by $U_\epsilon(x) = \epsilon^{\frac{4-N}{2}} U\left(\frac{x}{\epsilon}\right)$, where $\epsilon > 0$. The author showed the following results by using proper techniques.

Theorem 1.2.2. *([25]) Let λ_1 be the first eigenvalue of $-\Delta$ in $\Omega = B(0,1)$.*

(i) If $\lambda \in (0, \lambda_1)$ and $N \geq 8$, then problem (S_λ) has a nontrivial solution.

(ii) When $N = 5, 6$ or 7, then problem (S_λ) has a nontrivial solution for all $\lambda \in (\bar{\lambda}, \lambda_1)$, where $\bar{\lambda} = \lambda_1 - S|\Omega|^{-\frac{4}{N}}$.

(iii) When $N = 5, 6$ or 7 and $\Omega = B(0,1)$, then problem (S_λ) has a nontrivial solution for all $\lambda \in (\lambda_*, \lambda_1)$, where $\lambda_* = (\frac{\beta}{R})^4$, $\lambda_1 = (\frac{\alpha}{R})^4$ and α, β are two

positive numbers depending on N.

Thereafter, Shang and Li [61] studied the polyharmonic problem $(-\Delta)^m = \lambda f(x)|u|^{r-2} u + h(x)|u|^{2^*_m-2}u$, $1 < r < 2$, $N \geq 2m+1$, $2^*_m = \frac{2N}{N-2m}$ is the critical Sobolev exponent and f, h are sign-changing weight functions. Since the embedding $H^m_0(\Omega) \hookrightarrow L^{2^*_m}(\Omega)$ is not compact, so the corresponding energy functional does not satisfy the Palais-Smale condition in general. To overcome this difficulty, they studied the critical level with the help of best Sobolev constant S. Then it is well known that S is achieved if and only if $\Omega = \mathbb{R}^N$, by the function $U(x) = C_{N,m}^{\frac{N-2m}{4m}} (1+|x|^2)^{-\frac{N-2m}{2}}$ and all the minimizers of S are obtained by $U_\epsilon(x) = \epsilon^{\frac{2m-N}{2}} U\left(\frac{x}{\epsilon}\right)$, $\epsilon > 0$, where $C_{N,m} := C(N,m) = \prod_{j=1-m}^{m} (N-2j)$. Under some appropriate assumption, authors proved the following results in [61].

Theorem 1.2.3. *There exist $\Lambda_1 > 0$ and $\Lambda_2 > 0$ real constants such that.*

(i) The problem (S_λ) has at least one nontrivial solution for all $\lambda \in (0, \Lambda_1)$.

(ii) If $N > 4m$ and $\frac{N}{N-2m} \leq r < 2$, then there exists $\Lambda_2 > 0$ such that problem (S_λ) has at least two nontrivial solutions for all $\lambda \in (0, \Lambda_2)$.

Recently, Gao and Yang [32] studied the following Brézis-Nirenberg type critical problem for nonlinear Choquard equation

$$-\Delta u = \left(\int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy \right) |u|^{2^*_\mu - 2} u + \lambda u \ \text{ in } \ \Omega, u = 0 \ \text{ on } \ \partial\Omega, \tag{1.2.3}$$

where Ω is an open and bounded subset in $\mathbb{R}^N$ with Lipschitz boundary, $N \geq 3$, $2^*_\mu = \frac{2N-\alpha}{N-2}$, $\mu \in (0, N)$ and λ is a parameter. 2^*_μ is known as critical exponent in the sense of Hardy-Littlewood-Sobolev inequality.

To obtain the main results, they first proved the minimizer for $S_{H,L}$ in the case of Laplacian by using the function $U(x) = (N(N-2))^{\frac{N-2}{4}} (1+|x|^2)^{-\frac{N-2}{2}}$ and showed that $\bar{U}(x) = S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2}} C(N,\mu)^{\frac{2-N}{2(N-\mu+2}} (N(N-2))^{\frac{N-2}{4}} (1+|x|^2)^{-\frac{N-2}{2}}$ is a unique minimizer for $S_{H,L}$, where $S_{H,L}$ is defined as the best constant of Hardy-Littlewood-Sobolev inequality

$$S_{H,L} := \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu} dxdy\right)^{\frac{1}{2^*_\mu}}}.$$

Then by adopting the suitable technique and analyzing the Palais-Smale sequence below the critical level, they established the following results.

Theorem 1.2.4. *([32]) Let Ω is an open and bounded subset in $\mathbb{R}^N$ with Lipschitz boundary, $N \geq 3$ and $\mu \in (0, N)$. Then, the following results hold.*

(i) If $N \geq 4$, then the problem (1.2.3) has a nontrivial solution for $\lambda > 0$.

(ii) If $N = 3$, then there exists $\lambda_* > 0$ such that (1.2.3) has a nontrivial solution for $\lambda > \lambda_*$, where λ is not an eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary data.

With this, we note that there is no work on critical Choquard equation with biharmonic or p-biharmonic operator involving sign-changing weight functions. To tackle the sign-changing problems, Nehari manifold and fibering map technique are suitable tools which are introduced in next section.

1.3 Nehari manifold and fibering map technique

A substantial amount of work has been done to establish multiplicity results for semilinear, quasilinear elliptic equations with sign-changing nonlinearity by Nehari manifold and fibering map analysis, see [14, 16, 36, 59, 71] and references therein. So, we introduce the Nehari's method given by Zeev Nehari in 1960 ([56]). Let X be a real Banach space and $\mathcal{I}$ be a functional satisfying $\mathcal{I} \in C^1(X, \mathbb{R})$. Then we define a set, which contain all the critical points of $\mathcal{I}$, called Nehari manifold

$$\mathcal{N} := \{u \in X \setminus \{0\} : \langle \mathcal{I}'(u), u\rangle = 0\},$$

where $\mathcal{I}'$ denoted the Frechet derivative of $\mathcal{I}$. Generally $\mathcal{N}$ may not be manifold. Without loss of generality, we assume that $\mathcal{I}(0) = 0$. Take $G(u) := \mathcal{I}'(u)u$, then $G'(u)u = \mathcal{I}''(u)(u,u) + \mathcal{I}'(u)u \neq 0$ for each $0 \neq u \in \mathcal{N}$. Thus, implicit function theorem gives us that $\mathcal{N}$ is a C^1 manifold of codimension 1.

The Nehari manifold is closely related to the behavior of the mapping $\Phi_u : t \to \mathcal{I}(tu)$. These maps are known as fibering maps and given by Drabek and Pohozaev [24] and further discussed by Brown and Zhang [16]. It is easy to see that $\Phi_u'(t) = 0$ if and only if $tu \in \mathcal{N}$, i.e. $u \in \mathcal{N}$ iff $\Phi_u'(1) = 0$. Therefore, it is natural to split $\mathcal{N}$ into three parts $\mathcal{N}^+, \mathcal{N}^-$ and $\mathcal{N}^0$ corresponding to local minima, local maxima and point of inflection respectively. Thus, we have

$$\mathcal{N}^{\pm} := \{u \in \mathcal{N} : \Phi_u''(1) \gtrless 0\} \text{ and } \mathcal{N}^0 := \{u \in \mathcal{N} : \Phi_u''(1) = 0\}.$$

Therefore, for the existence of nontrivial solutions, the main essence is to minimize $\mathcal{I}$ on a suitable subset of $\mathcal{N}$ and show that these minimizers are the critical points of corresponding energy functional $\mathcal{I}$.

The following Lemma shows that the local minimizer of $\mathcal{I}$ on $\mathcal{N}^+$ or $\mathcal{N}^-$ becomes the critical point of $\mathcal{I}$.

Lemma 1.3.1. *If u is the local minimizer for $\mathcal{I}$ on subsets $\mathcal{N}^+$ or $\mathcal{N}^-$ of $\mathcal{N}$ such that $u \notin \mathcal{N}^0$. Then $\mathcal{I}'(u) = 0$ in X^{-1}, where X^{-1} denotes the dual space of X.*

Proof. Let u be a local minimizer for $\mathcal{I}$ subject to the constrains $\Psi(u) := \langle \mathcal{I}'(u), u\rangle = 0$. Then by Lagrange multipliers, there exists $\theta \in \mathbb{R}$ such that $\mathcal{I}'(u) = \theta\Psi'(u)$. Thus $\langle \mathcal{I}'(u), u\rangle = \theta\langle \Psi'(u), u\rangle$. Since $u \in \mathcal{N}$, then $\langle \mathcal{I}'(u), u\rangle = 0$ and $\langle \Psi'(u), u\rangle \neq 0$, since $u \notin \mathcal{N}^0$. Hence, $\theta = 0$ which completes the proof. □

1.4 Structure of Thesis

The thesis comprises six chapters. In chapter 1, we give a brief introduction about operators, embeddings, Choquard nonlinearity, Nehari manifold and fibering map analysis which play a vital role to study the problems. In chapter 2, we deal with biharmonic equation with critical Choquard type nonlinearity having sign-changing weight functions. We show the minimizers for $S_{H,L}$ corresponding to Choquard equation involving biharmonic operator and using these minimizers, we obtain the existence and multiplicity results of nontrivial solution with respect to parameter $\lambda > 0$. Later, we extend this work to the biharmonic system with Hartree type nonlinearity in chapter 3. In chapter 4, we study p-biharmonic Choquard equations with subcritical or critical nonlinearity and prove the multiplicity results of solutions in subcritical case and existence of a solution in critical case. In chapter 5, we concern with the existence of multiple solutions to the polyharmonic system involving concave-convex nonlinearities with critical exponent and sign-changing weight functions. In chapter 6, we give the summary of conclusions and provide an overview of some future work.

1.4.1 Critical biharmonic equation with Choquard nonlinearity

In chapter 2, we consider the following biharmonic Choquard equation with critical exponent and sign-changing weight functions

$$(E_\lambda)\begin{cases} \Delta^2 u = \lambda f(x)|u|^{q-2}u + g(x)\left(\displaystyle\int_\Omega \frac{g(y)|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dy\right)|u|^{2^*_\alpha-2}u & \text{in } \Omega,\\ u, \nabla u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 5$, $1 < q < 2$, $0 < \alpha < N$, $2^*_\alpha = \frac{2N-\alpha}{N-4}$ is the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality, Δ^2 denotes the biharmonic operator and $\lambda > 0$ is a parameter. The functions $f, g : \bar{\Omega} \to \mathbb{R}$ are sign-changing continuous functions satisfying the following:

$(f1)$ $f \in C(\overline{\Omega})$ and $f^+ = \max\{f, 0\} \not\equiv 0$ in Ω.

$(g1)$ $g \in C(\overline{\Omega})$ and $g^+ = \max\{g, 0\} \not\equiv 0$ in Ω.

$(f2)$ There exist a_0 and $r_0 > 0$ such that $B(0, 2r_0) \subset \Omega$ and $f(x) \geq a_0$ for all $x \in B(0, 2r_0)$.

$(g2)$ There exists $\delta_0 > \frac{2N-\alpha}{2}$ such that

$$g(x) = g(0) + o(|x|^{\delta_0}), \quad \text{as } x \to 0$$

and $\|g^+\|_\infty = g(0) = \max_{x\in\bar{\Omega}} g(x),\ g(x) > 0$ for all $x \in B(0, 2r_0)$.

Set

$$\Gamma_1 := \left[\left(\frac{2-q}{22^*_\alpha - q}\right)\|g^+\|_\infty^{-1} S_{H,L}^{2^*_\alpha}\right]^{\frac{2-q}{22^*_\alpha-2}} \left(\frac{22^*_\alpha - 2}{22^*_\alpha - q}\right)\|f^+\|_\infty^{-1}|\Omega|^{\frac{q-2^*_2}{2^*_2}} S^{\frac{q}{2}}.$$

Theorem 1.4.1. *Suppose the assumptions $(f1)$ and $(g1)$ hold. Then, there exists $\Gamma_1 > 0$ such that the problem (E_λ) has at least a nontrivial solution for every $\lambda \in (0, \Gamma_1)$.*

Theorem 1.4.2. *Suppose that $(f1), (g1), (f2)$ and $(g2)$ hold. Then there exists $\Gamma_2 > 0$, such that the problem (E_λ) has at least two nontrivial solutions for every $\lambda \in (0, \Gamma_2)$, where $\Gamma_2 = \min\{\bar{\Lambda}, \frac{q}{2}\Gamma_1\}$ with $\bar{\Lambda} > 0$ a constant.*

We first define the energy functional corresponding to the problem (E_λ) as

$$\mathcal{I}_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2.2^*_\alpha}\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy - \frac{\lambda}{q}\int_\Omega f(x)|u|^q dx.$$

Then $\mathcal{I}_\lambda \in C^1(H_0^2(\Omega), \mathbb{R})$, by Hardy-Littlewood-Sobolev inequality. Moreover, u is a weak solution of the problem (E_λ) if and only if u is a critical point of the corresponding energy functional $\mathcal{I}_\lambda$.

Since the energy functional $\mathcal{I}_\lambda$ is unbounded below on $H_0^2(\Omega)$. So we restrict $\mathcal{I}_\lambda$ to a suitable subset $\mathcal{N}_\lambda$ of $H_0^2(\Omega)$, called Nehari set and defined as

$$\mathcal{N}_\lambda := \left\{u \in H_0^2(\Omega) \setminus \{0\} : \langle \mathcal{I}'_\lambda(u), u\rangle = 0\right\}.$$

We note that $u \in \mathcal{N}_\lambda$ if and only if

$$\langle \mathcal{I}'_\lambda(u), u\rangle = \|u\|^2 - \int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy - \lambda\int_\Omega f(x)|u|^q dx = 0.$$

It means that $\mathcal{N}_\lambda$ contains every nontrivial solution of (E_λ). $\mathcal{I}_\lambda$ is coercive and bounded below on $\mathcal{N}_\lambda$. Then we define the fiber maps $\Phi_u : t \to \mathcal{I}_\lambda(tu)$ as

$$\Phi_u(t) = \mathcal{I}_\lambda(tu) = \frac{t^2}{2}\|u\|^2 - \frac{t^{22^*_\alpha}}{22^*_\alpha}\int_\Omega\int_\Omega g(x)\frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy - \frac{\lambda t^q}{q}\int_\Omega f(x)|u|^q dx.$$

It is easy to see that $\Phi'_u(t) = 0$ if and only if $tu \in \mathcal{N}_\lambda$, i.e. $u \in \mathcal{N}_\lambda$ iff $\Phi'_u(1) = 0$. Moreover,

$$\mathcal{N}_\lambda^\pm := \{u \in \mathcal{N}_\lambda : \Phi''_u(1) \gtrless 0\} \text{ and } \mathcal{N}_\lambda^0 := \{u \in \mathcal{N}_\lambda : \Phi''_u(1) = 0\},$$

where

$$\Phi''_u(1) = \begin{cases} (2-2.2^*_\alpha)\|u\|^2 - \lambda(q-2.2^*_\alpha)\int_\Omega f(x)|u|^q dx \\ (2-q)\|u\|^2 - (2.2^*_\alpha - q)\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy. \end{cases}$$

The following lemma is a consequence of $\mathcal{N}_\lambda$ is a C^1 manifold.

Lemma 1.4.1. *If u is the local minimizer for $\mathcal{I}_\lambda$ on subsets $\mathcal{N}_\lambda^+$ or $\mathcal{N}_\lambda^-$ of $\mathcal{N}_\lambda$ such that $u \notin \mathcal{N}_\lambda^0$, then u becomes the critical point for $\mathcal{I}_\lambda$.*

By analysing the fiber maps, we have the following lemma:

Lemma 1.4.2. *Suppose $0 < \lambda < \Gamma_1$ and $u \in H_0^2(\Omega)$. If $\int_\Omega f(x)|u|^q dx > 0$ and $\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha} > 0$, then there exists unique t^+ and t^- satisfying $0 < t^+ < t_{\max} < t^-$ such that $t^+(u)u \in \mathcal{N}_\lambda^+$ and $t^-(u)u \in \mathcal{N}_\lambda^-$. Moreover*

$$\mathcal{I}_\lambda(t^+u) = \inf_{0 \le t \le t_{\max}} \mathcal{I}_\lambda(tu);\ \mathcal{I}_\lambda(t^-u) = \sup_{t \ge t_{\max}} \mathcal{I}_\lambda(tu).$$

Moreover, we conclude that $\mathcal{N}_\lambda^0 = \phi$ for $0 < \lambda < \Gamma_1$ and we study the existence of minimizers u_λ and v_λ of $\mathcal{I}_\lambda$ in $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ respectively. Thus, with a chain of lemmas, we prove that the minimizing sequence $\{u_n\}$ of $\mathcal{I}_\lambda$ is a Palais-Smale sequence and weak limit of $\{u_n\}$ i.e. u_λ is a minimizer of $\mathcal{I}_\lambda$ in $\mathcal{N}_\lambda^+$ which is characterized as the first solution to the problem (E_λ).

To obtain the second solution, we first study the minimizers for $S_{H,L}$ and show the relation between S and $S_{H,L}$, where $S_{H,L}$ is the best constant and defined as

$$S_{H,L} := \inf_{u \in D^{2,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\Delta u|^2 dx}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{2^*_\alpha}}}.$$

Take $\widehat{U}_\epsilon(x) = S^{\frac{(N-\alpha)(4-N)}{8(N+4-\alpha)}} (C(N,\alpha))^{\frac{4-N}{2(N+4-\alpha)}} U_\epsilon(x)$, where $\epsilon > 0$, then $\widehat{U}_\epsilon$ gives a family of minimizers for $S_{H,L}$ and satisfies the equation

$$\Delta^2 u = \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\alpha}}{|x-y|^\alpha} dy\right) |u|^{2^*_\alpha - 2} u \quad \text{ in } \mathbb{R}^N.$$

Hence, with the help of sequence of Lemmas and Theorems, we find that, for the existence of second solution in critical case, one needs to study the critical level (c_∞). Since the embedding $H_0^2(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact. Due to the lack of compactness, one can not pass the limit in the minimizing sequence. Further, the local compactness is recovered by studying the critical level below which Palais-Smale sequences are compact and the critical level is calculated with the help of minimizers of $S_{H,L}$. Weak limit of Palais-Smale sequence i.e. v_λ is a minimizer of $\mathcal{I}_\lambda$ in $\mathcal{N}_\lambda^-$ and becomes the second solution of (E_λ).

1.4.2 Biharmonic system with critical Hartree-type nonlinearity

In chapter 3, we concern with the following biharmonic Choquard system involving critical nonlinearities with sign-changing weight function:

$$(\mathcal{D}_{\lambda,\mu}) \begin{cases} \Delta^2 u = \lambda F(x)|u|^{r-2}u + H(x)\left(\int_\Omega \frac{H(y)|v(y)|^{2^*_\alpha}}{|x-y|^\alpha} dy\right)|u|^{2^*_\alpha-2}u & \text{in } \Omega, \\ \Delta^2 v = \mu G(x)|v|^{r-2}v + H(x)\left(\int_\Omega \frac{H(y)|u(y)|^{2^*_\alpha}}{|x-y|^\alpha} dy\right)|v|^{2^*_\alpha-2}v & \text{in } \Omega, \\ u = v = \nabla u = \nabla v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 5$, $1 < r < 2$, $0 < \alpha < N$, $2^*_\alpha = \frac{2N-\alpha}{N-4}$ is the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality. We prove the existence and multiplicity results of the nontrivial solutions for system with respect to parameter $(\lambda, \mu) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$ by adopting Nehari manifold and fibering map technique.

The novelty of this work is that the results obtained here are also new for Laplacian case also. Precisely, we extend the problem (E_λ) to the biharmonic system of equations $(\mathcal{D}_{\lambda,\mu})$. Altogether, this work amplifies the branch of knowledge and give a novel addition in the literature of critical Choquard system. With this contribution in the literature of system with critical Choquard type nonlinearity, we assume the following assumptions on the weight functions F, G and H to tackle the problem:

- $(Z1)$ $F, G \in L^\beta(\Omega)$ with $\beta = \frac{2^*}{2^*-r}$ and $2^* = \frac{2N}{N-4}$, $F^\pm = \max\{\pm F, 0\} \not\equiv 0$ in $\overline{\Omega}$ and $G^\pm = \max\{\pm G, 0\} \not\equiv 0$ in $\overline{\Omega}$.
- $(Z2)$ $H \in L^\infty(\Omega)$ and $H^+ = \max\{H, 0\} \not\equiv 0$ in Ω.

The main result is as follows.

Theorem 1.4.3. *If $1 \leq r < 2$, $0 < \alpha < N$ and $\lambda, \mu > 0$ satisfy $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$ (where $\Upsilon_1 > 0$ constant), then the system $(\mathcal{D}_{\lambda,\mu})$ has at least one nontrivial solution in $H^2_0(\Omega) \times H^2_0(\Omega)$.*

In order to obtain the second nontrivial solution of $(\mathcal{D}_{\lambda,\mu})$, the following assumptions are imposed on functions F, G and H:

- $(Z3)$ There exist a_0, b_0 and $r_0 > 0$ such that $B(0, 2r_0) \subset \Omega$ and $F(x) \geq a_0$, $G(x) \geq b_0$ for all $x \in B(0, 2r_0)$.
- $(Z4)$ There exists $\delta_0 > \frac{2N-\alpha}{2}$ such that $\|H^+\|_\infty = H(0) = \max_{x\in\overline{\Omega}} h(x)$, $H(x) > 0$ for all $x \in B(0, 2r_0)$ and

$$H(x) = H(0) + o\left(|x|^{\delta_0}\right) \text{ as } x \to 0.$$

Theorem 1.4.4. *If $1 \leq r < 2$, $0 < \alpha < N$ and $\lambda, \mu > 0$ satisfy $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_2$ (where $\Upsilon_2 \leq \Upsilon_1$) , then the system $(\mathcal{D}_{\lambda,\mu})$ has at least two non-trivial solution in $H^2_0(\Omega) \times H^2_0(\Omega)$.*

For the existence of a second solution in critical, we study the critical level (c_∞) below which Palais-Smale sequences are compact and the critical level is calculated with the help of minimizers of $S_{H,L}$, where

$$c_\infty := \frac{N+4-\alpha}{2(2N-\alpha)}\left(\frac{\|H^+\|_\infty^{-2}}{2}\right)^{\frac{N-4}{N+4-\alpha}} \bar{S}_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}} - K_0\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right),$$

with $K_0 > 0$ constant. Moreover the following estimate plays a crucial role to show that minimum of $I_{\lambda,\mu}$ is attained over $\mathcal{N}_{\lambda,\mu}^-$.

Lemma 1.4.3. *For Choquard term, the following estimate is true:*

$$\begin{aligned} 0 &\le \|H^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}} (C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}} S_{H,L}^{\frac{N-4}{4}} - o\left(\epsilon^{\frac{2N-\alpha}{2}}\right) \\ &\le \left(\int_\Omega\int_\Omega H(x)H(y)\frac{|\overline{U}_\epsilon(x)|^{2_\alpha^*}|\overline{U}_\epsilon(y)|^{2_\alpha^*}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2_\alpha^*}} \le \|H^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}} (C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}} S_{H,L}^{\frac{N-4}{4}}. \end{aligned}$$

1.4.3 p-biharmonic equation involving Choquard nonlinearity

In chapter 4, we consider the following p-biharmonic equation with nonlocal term

$$(\mathcal{G}_\lambda)\begin{cases} \Delta_p^2 u = \lambda f(x)|u|^{r-2}u + g(x)\left(\displaystyle\int_\Omega \frac{g(y)|u(y)|^q}{|x-y|^\alpha}dy\right)|u|^{q-2}u & \text{in } \Omega, \\ u, \nabla u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain having smooth boundary $\partial\Omega$, $p \in (1,\infty)$, $N > 2p$, $1 < r < p$, $0 < \alpha < N$, $\frac{p(2N-\alpha)}{2N} \le q \le \frac{p(2N-\alpha)}{2(N-2p)}$, $p_\alpha^* := \frac{p(2N-\alpha)}{2(N-2p)}$ is the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality and $\lambda > 0$ is a parameter. On applying Nehari manifold technique and concentration-compactness principle, we show the multiplicity results in subcritical case and existence results in critical case with respect to parameter λ.

To examine the problem $(\mathcal{G}_\lambda)$, the main difficulty is lack of compactness in the embeddings of $W_0^{2,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$. To handle this complication, concentration-compactness principles [50, 51] are used, given by P. L. Lions and we are adequate to prove the local Palais-Smale condition. Furthermore, in our awareness, there is no appearance of such Lemma that elaborate the concentration of a weakly convergent sequence at finite points in case of p-biharmonic critical Choquard equation. So our work delivers a contribution in the literature of p-biharmonic equation with Choquard nonlinearity.

To prove our main results, we require some assumptions on sign-changing weight functions f and g respectively. So precisely describing, our assumptions are:

$(f1)$ $f \in L^{\beta}(\Omega)$, where $\beta = \frac{p^*}{p^*-r}$ with $p^* = \frac{Np}{N-2p}$ and $f^+ = \max\{f, 0\} \not\equiv 0$ in Ω.

$(g1)$ $g \in C(\overline{\Omega})$ and $g^+ = \max\{g, 0\} \not\equiv 0$ in Ω.

We state our following main results.

Theorem 1.4.5. *Suppose the assumptions $(f1)$ and $(g1)$ hold. If $1 < r < p$ and $\frac{p(2N-\alpha)}{2N} \leq q < p^*_\alpha$, then there exists $\Upsilon_1 > 0$ such that the problem $(\mathcal{G}_\lambda)$ has at least one nontrivial solution in $\mathcal{N}^+_\lambda$ for every $\lambda \in (0, \Upsilon_1)$.*

Theorem 1.4.6. *Suppose the assumptions $(f1)$ and $(g1)$ hold. If $1 < r < p$ and $\frac{p(2N-\alpha)}{2N} \leq q < p^*_\alpha$, then there exists $\Upsilon_2 > 0$ such that the problem $(\mathcal{G}_\lambda)$ has at least one nontrivial solution in $\mathcal{N}^-_\lambda$ for every $\lambda \in (0, \Upsilon_2)$.*

To obtain the existence results in critical case, we first prove the following concentration-compactness principle which plays a crucial role to tackle the problem $(\mathcal{G}_\lambda)$. Idea of the proof is taken from [50, 51].

Lemma 1.4.4. *Assume that $\{u_n\}$ be a bounded sequence in $D^{2,p}(\mathbb{R}^N)$ converging weakly and a.e. to $u \in D^{2,p}(\mathbb{R}^N)$. Suppose*

$$\left(\int_\Omega \frac{|u_n(y)|^{p^*_\alpha}}{|x-y|^\alpha} dy\right) |u_n(x)|^{p^*_\alpha} \rightharpoonup \eta,$$

in the sense of measure. Then there exists a countable sequence of points $\{x_i\}_{i \in J} \subset \mathbb{R}^N$ and families of positive numbers $\{\mu_i : i \in J\}$, $\{\omega_i : i \in J\}$ and $\{\eta_i : i \in J\}$ such that

$$\eta = \left(\int_\Omega \frac{|u(y)|^{p_\alpha^*}}{|x-y|^\alpha}dy\right)|u(x)|^{p_\alpha^*} + \sum_{i\in J}\eta_i\delta_{x_i},$$
$$\mu \geq |\Delta u|^p + \sum_{i\in J}\mu_i\delta_{x_i}, \omega \geq |u|^{p^*} + \sum_{i\in J}\omega_i\delta_{x_i},$$
$$S_{H,L}\eta_i^{\frac{N-2p}{2N-\alpha}} \leq \mu_i,\ \eta_i \leq C(N,\alpha)\omega_i^{\frac{2N-\alpha}{N}},$$

where η, μ and ω are bounded and nonnegative measures on $\mathbb{R}^N$ and δ_{x_i} is the Dirac mass at x_i.

Theorem 1.4.7. *Suppose that assumptions $(f1)$ and $(g1)$ hold. If $1 < r < p$ and $\frac{p(2N-\alpha)}{2N} \leq q = p_\alpha^*$, then there exists $\Upsilon_1 > 0$ such that the problem $(\mathcal{G}_\lambda)$ has at least one nontrivial solution in $\mathcal{N}_\lambda^+$ for every $\lambda \in (0, \Upsilon_1)$.*

1.4.4 Polyharmonic system with concave-convex nonlinearities

In chapter 5, we investigate the existence of multiple solutions of the following polyharmonic system involving concave-convex nonlinearities with critical exponent and sign-changing weight functions

$$(E_{\lambda,\mu}) \begin{cases} (-\Delta)^m u = \lambda f(x)|u|^{r-2}u + \frac{\beta}{\beta+\gamma}h(x)|u|^{\beta-2}u|v|^\gamma & \text{in } \Omega, \\ (-\Delta)^m v = \mu g(x)|v|^{r-2}v + \frac{\gamma}{\beta+\gamma}h(x)|u|^\beta|v|^{\gamma-2}v & \text{in } \Omega, \\ D^k u = D^k v = 0 \quad \text{for all} \quad |k| \leq m-1 & \text{on } \partial\Omega, \end{cases}$$

where Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $m \in \mathbb{N}$, $N \geq 2m+1$, $1 < r < 2$, $\beta > 1$, $\gamma > 1$ satisfying $2 < \beta+\gamma \leq 2_m^*$ with $2_m^* = \frac{2N}{N-2m}$ as a critical Sobolev exponent, Δ^m denotes the polyharmonic operators and λ, μ are the parameter such that $(\lambda,\mu) \in \mathbb{R}_+^2 \setminus \{(0,0)\}$.
The motivation behind the study of polyharmonic system involving critical nonlinearity and sign-changing weight functions is the work of Shang and Li [61]. They investigated the multiplicity of nontrivial solutions of the problem $(E_{\lambda,\mu})$ in the case

of $\beta = \gamma$, $\beta + \gamma = 2^*_m$, $\lambda = \mu$, $u = v$ and $f \equiv g$. Furthermore, there is no result so far concerning polyharmonic system involving critical nonlinearities with sign-changing weight functions. Apart from this, the results obtained here are even new for linear case ($m = 2$).

Since the embedding $H_0^m(\Omega) \hookrightarrow L^{2^*_m}(\Omega)$ is not compact, so the corresponding energy functional associated to the problem does not satisfy the Palais-Smale condition in general. Therefore, it is difficult to obtain the critical points of energy functional by simple arguments, which are based on the compactness of the Sobolev embedding. To overcome this difficulty, we extract a Palais-Smale sequence in the Nehari manifold and show that the Palais-Smale sequence satisfies $(PS)_c$-condition below the critical level, which is studied explicitly using the minimizers and we conclude that weak limit is the required solution.

To obtain our results, we need the following additive assumptions on the weight functions f, g and h:

- $(a1)$ $f,\ g \in L^\alpha(\Omega)$ with $\alpha = \frac{\beta+\gamma}{\beta+\gamma-r}$, $f^\pm = \max\{\pm f, 0\} \not\equiv 0$ in $\overline{\Omega}$ and $g^\pm = \max\{\pm g, 0\} \not\equiv 0$ in $\overline{\Omega}$ i.e. (f and g are possibly sign-changing on $\overline{\Omega}$).
- $(h1)$ $h \in L^\infty(\Omega)$ and $h^+ = \max\{h, 0\} \not\equiv 0$ in Ω.
- $(a2)$ There exist a_0, b_0 and $r_0 > 0$ such that $B(0, 2r_0) \subset \Omega$ and $f(x) \geq a_0$, $g(x) \geq b_0$ for all $x \in B(0, 2r_0)$.
- $(h2)$ There exists $\delta_0 > 0$ such that $\|h\|_\infty = h(0) = \max\limits_{x\in\overline{\Omega}} h(x)$, $h(x) > 0$ for all $\in B_{2r_0}(0)$ and
$$h(x) = h(0) + o(|x|^{\delta_0}) \text{ as } x \longrightarrow 0.$$

Now, we state the results.

Theorem 1.4.8. *Assume that* $(a1)$, $(h1)$ *hold. If* $1 \leq r < 2 < \frac{N}{m}$, $2 < \beta + \gamma \leq 2^*_m$, *and* $\lambda, \mu > 0$ *satisfy* $0 < (\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} < \Lambda_1$, *then the system* $(E_{\lambda,\mu})$ *has at least one nontrivial solution in* $H_0^m(\Omega) \times H_0^m(\Omega)$.

Theorem 1.4.9. *(Second nontrivial solution in subcritical case). Assume that* $(a1)$, $(h1)$ *hold. If* $1 \leq r < 2 < \frac{N}{m}$, $2 < \beta + \gamma < 2^*_m$, *and* $\lambda, \mu > 0$ *satisfy* $0 < (\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} < (\frac{r}{2})^{\frac{2}{2-r}}\Lambda_1$, *then the system* $(E_{\lambda,\mu})$ *has at least two nontrivial solution* *in* $H_0^m(\Omega) \times H_0^m(\Omega)$.

Theorem 1.4.10. *(Second nontrivial solution in critical case). Assume that* $(a1)-(h2)$ *hold. If* $1 \leq r < 2 < \frac{N}{m}$*, and* $\lambda, \mu > 0$ *satisfy* $0 < (\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} < (\frac{r}{2})^{\frac{2}{2-r}}\Lambda_1$ *, then the system* $(E_{\lambda,\mu})$ *has at least two nontrivial solution in* $H_0^m(\Omega) \times H_0^m(\Omega)$.

Where

$$\Lambda_1 := \left(\frac{2-r}{(\beta+\gamma-r)\|h\|_\infty}\right)^{\frac{2}{\beta+\gamma-2}} \left(\frac{\beta+\gamma-r}{\beta+\gamma-2}\right)^{\frac{-2}{2-r}} S^{\frac{2(\beta+\gamma-r)}{(2-r)(\beta+\gamma-2)}} > 0.$$

1.4.5 Conclusion and future work

In chapter 6, we give the conclusions of our research work covered in this thesis and provide the direction for some future work.

2

Biharmonic Equation With Choquard Nonlinearity Involving Sign-Changing Weight Functions

In this chapter, we are concerned with the existence and multiplicity results of nontrivial solutions for biharmonic critical Choquard equation involving sign-changing weight functions with respect to the parameter λ. In this direction, Gao and Yang [32] studied the critical problem for nonlinear Choquard equation with Laplacian and showed the existence and nonexistence of the nontrivial solution. Further, Mukherjee and Sreenadh [54] studied the fractional Laplacian equations with critical Choquard nonlinearity.

2.1 Inroduction to the problem

Consider the following biharmonic critical Choquard equation

$$(E_\lambda)\begin{cases} \Delta^2 u = \lambda f(x)|u|^{q-2}u + g(x)\left(\displaystyle\int_\Omega \frac{g(y)|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dy\right)|u|^{2^*_\alpha-2}u & \text{in } \Omega,\\ u, \nabla u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 5$, $1 < q < 2$, $0 < \alpha < N$, $2^*_\alpha = \frac{2N-\alpha}{N-4}$ is the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality, Δ^2 denotes the biharmonic operator and $\lambda > 0$ is a parameter. Before stating our main results, we assume the following conditions on the functions f and g. The functions $f, g : \bar{\Omega} \to \mathbb{R}$ are sign-changing continuous functions satisfying the following:

$(f1)$ $f \in C(\overline{\Omega})$ and $f^+ = \max\{f, 0\} \not\equiv 0$ in Ω.

$(g1)$ $g \in C(\overline{\Omega})$ and $g^+ = \max\{g, 0\} \not\equiv 0$ in Ω.

Set

$$\Gamma_1 = \left[\left(\frac{2-q}{2.2^*_\alpha - q}\right)\|g^+\|_\infty^{-2} S_{H,L}^{2^*_\alpha}\right]^{\frac{2-q}{2.2^*_\alpha-2}}\left(\frac{2.2^*_\alpha - 2}{2.2^*_\alpha - q}\right)\|f^+\|_\infty^{-1}|\Omega|^{\frac{q-2^*}{2^*}} S^{\frac{q}{2}}. \qquad (2.1.1)$$

Theorem 2.1.1. *Suppose the assumptions $(f1)$ and $(g1)$ hold. Then there exists $\Gamma_1 > 0$ such that (E_λ) has at least a nontrivial solution for every $\lambda \in (0, \Gamma_1)$.*

In order to obtain a second nontrivial solution of (E_λ), we need the following additional assumptions on the functions f and g:

$(f2)$ There exist a_0 and $r_0 > 0$ such that $B(0, 2r_0) \subset \Omega$ and $f(x) \geq a_0$ for all $x \in B(0, 2r_0)$.

$(g2)$ There exists $\delta_0 > \frac{2N-\alpha}{2}$ such that

$$g(x) = g(0) + o(|x|^{\delta_0}), \quad \text{as } x \to 0$$

and $\|g^+\|_\infty = g(0) = \max_{x\in\bar{\Omega}} g(x),\ g(x) > 0$ for all $x \in B(0, 2r_0)$.

Theorem 2.1.2. *Suppose that* $(f1), (g1), (f2),$ *and* $(g2)$ *hold. Then the problem* (E_λ) *has at least two nontrivial solutions for every* $\lambda \in (0, \Gamma_2)$*, where* $\Gamma_2 = \min\{\bar{\Lambda}, \frac{q}{2}\Gamma_1\}$ *with* $\bar{\Lambda} > 0$ *constant.*

Definition 2.1.1. *A function* $u \in H_0^2(\Omega)$ *is said to be a weak solution of* (E_λ) *if, for all* $\phi \in H_0^2(\Omega)$,

$$\int_\Omega \Delta u \Delta \phi dx - \lambda \int_\Omega f(x)|u|^{q-2}u\phi dx - \int_\Omega \int_\Omega g(x)g(y)\frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha-2}u(y)\phi(y)}{|x-y|^\alpha}dxdy = 0.$$

Now, we define the associated energy functional corresponding to the problem (E_λ) as

$$\mathcal{I}_\lambda(u) = \frac{\|u\|^2}{2} - \frac{1}{2.2^*_\alpha}\int_\Omega \int_\Omega \frac{g(x)g(y)|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy - \frac{\lambda}{q}\int_\Omega f(x)|u|^q. \quad (2.1.2)$$

Then $\mathcal{I}_\lambda \in C^1(H_0^2(\Omega), \mathbb{R})$, by Hardy-Littlewood-Sobolev inequality (1.1.2) and

$$\langle \mathcal{I}'_\lambda(u), \phi \rangle = \int_\Omega \Delta u \Delta \phi - \int_\Omega \int_\Omega \frac{g(x)g(y)|u(y)|^{2^*_\alpha}|u(x)|^{2^*_\alpha-1}\phi(x)}{|x-y|^\alpha}dxdy - \lambda \int_\Omega f(x)|u|^{q-1}\phi.$$

One can easily see that u is a weak solution to the problem (E_λ) if and only if u is a critical point of the functional $\mathcal{I}_\lambda$.

2.2 Minimizers and technical lemmas

In this section, we show the minimizers corresponding to Hardy-Littlewood-Sobolev inequality for biharmonic operator and prove some technical Lemmas that are used to get the desirable results.
From the Hardy-Littlewood-Sobolev inequality (1.1.2), the integral $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^s|u(y)|^s}{|x-y|^\alpha}dxdy$ is well defined if $|u|^s \in L^t(\mathbb{R}^N)$ for $t > 1$ such that $\frac{2}{t} + \frac{\alpha}{N} = 2$. Thus for $u \in H^2(\mathbb{R}^N)$, using Sobolev embedding theorems, we obtain

$$2_\alpha := \frac{2N-\alpha}{N} \leq s \leq \frac{2N-\alpha}{N-4} := 2^*_\alpha,$$

where $2_\alpha := \frac{2N-\alpha}{N}$ and $2^*_\alpha = \frac{2N-\alpha}{N-4}$ are known as lower and upper critical exponent respectively in the sense of Hardy-Littlewood-Sobolev inequality. We define $S_{H,L}$ to be the best constant as

$$S_{H,L} := \inf_{u\in D^{2,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\Delta u|^2 dx}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{2^*_\alpha}}}. \tag{2.2.3}$$

Now, for all $u \in D^{2,2}(\mathbb{R}^N)$, by Hardy-Littlewood-Sobolev inequality (1.1.2), one can easily deduce that

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{2^*_\alpha}} \leq (C(N,\alpha))^{\frac{1}{2^*_\alpha}} \|u\|^2_{2^*}, \tag{2.2.4}$$

where $C(N,\alpha)$ is same as defined in Proposition 1.1.1. Now using this together with the definition of $S_{H,L}$ yield

$$S_{H,L}(C(N,\alpha))^{\frac{1}{2^*_\alpha}} \geq S. \tag{2.2.5}$$

We notice that the equality holds in the inequality (1.1.2) if and only if

$$h(x) = C\left(\frac{k}{k^2+|x-a|^2}\right)^{\frac{2N-\alpha}{2}},$$

where $C > 0$ is fixed constant. Therefore, $u = C\left(\frac{k}{k^2+|x-a|^2}\right)^{\frac{N-4}{2}}$ if and only if

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{2^*_\alpha}} = (C(N,\alpha))^{\frac{1}{2^*_\alpha}} \left(\int_{\mathbb{R}^N} |u(x)|^{2^*} dx\right)^{\frac{2}{2^*}}. \tag{2.2.6}$$

Further we show the relation between S and $S_{H,L}$. The main idea is taken from [32].

Theorem 2.2.1. *The constant $S_{H,L}$ is achieved if and only if $u = C\left(\frac{k}{k^2+|x-a|^2}\right)^{\frac{N-4}{2}}$, where $C > 0$ is a constant, $a \in \mathbb{R}^N$ and $k \in \mathbb{R}^+$. Moreover*

$$S_{H,L}(C(N,\alpha))^{\frac{1}{2^*_\alpha}} = S. \tag{2.2.7}$$

Proof. With the help of definition of $S_{H,L}$ and (2.2.6), we obtain

$$S_{H,L} \leq \frac{\int_{\mathbb{R}^N} |\Delta u|^2 dx}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}\right)^{\frac{1}{2^*_\alpha}}} \leq \frac{\int_{\mathbb{R}^N} |\Delta u|^2 dx}{(C(N,\alpha))^{\frac{1}{2^*_\alpha}} (\int_{\mathbb{R}^N} |u(x)|^{2^*} dx)^{\frac{2}{2^*}}} \leq \frac{S}{(C(N,\alpha))^{\frac{1}{2^*_\alpha}}}.$$

This together with (2.2.5), we obtain $S_{H,L}(C(N,\alpha))^{\frac{1}{2^*_\alpha}} = S$. Thus, we observe that $S_{H,L}$ is achieved if and only if $u = C\left(\frac{k}{k^2+|x-a|^2}\right)^{\frac{N-4}{2}}$ and hence

$$S_{H,L}(C(N,\alpha))^{\frac{1}{2^*_\alpha}} = S.$$

This completes the proof. □

In particular, consider for $\epsilon > 0$, $\widehat{U}_\epsilon(x) = S^{\frac{(N-\alpha)(4-N)}{8(N+4-\alpha)}}(C(N,\alpha))^{\frac{4-N}{2(N+4-\alpha)}}U_\epsilon(x)$, where

$$U_\epsilon(x) = \epsilon^{\frac{4-N}{2}} U\left(\frac{x}{\epsilon}\right) \text{ with } U(x) = \frac{[N(N+2)(N-2)(N-4)]^{\frac{N-4}{8}}}{(1+|x|^2)^{\frac{N-4}{2}}}, \tag{2.2.8}$$

then $\widehat{U}_\epsilon$ gives a family of minimizers for $S_{H,L}$ and satisfies the equation

$$\Delta^2 u = \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\alpha}}{|x-y|^\alpha} dy\right)|u|^{2^*_\alpha - 2}u \quad \text{ in } \mathbb{R}^N.$$

Moreover,

$$\int_{\mathbb{R}^N} |\Delta \widehat{U}_\epsilon|^2 dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\widehat{U}_\epsilon(x)|^{2^*_\alpha}|\widehat{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy = (S_{H,L})^{\frac{2N-\alpha}{N+4-\alpha}}.$$

Lemma 2.2.1. *Let $N \geq 5$ and Ω be an open subset of $\mathbb{R}^N$, then*

$$S_{H,L}(\Omega) := \inf_{u \in H_0^2(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta u|^2 dx}{\left(\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{2^*_\alpha}}} = S_{H,L},$$

that is, $S_{H,L}(\Omega)$ is achieved only when $\Omega = \mathbb{R}^N$.

Proof. It is observed that $S_{H,L} \leq S_{H,L}(\Omega)$. Let $\{u_n\} \subset C_c^\infty(\mathbb{R}^N)$ be a minimizing sequence for $S_{H,L}$. Then we choose $z_n \in \mathbb{R}^N$ and $\kappa_n > 0$ such that $v_n(x) = \kappa_n^{\frac{N-4}{2}} u_n(\kappa_n x + z_n) \in C_c^\infty(\Omega)$. Now v_n satisfies

$$\int_{\mathbb{R}^N} |\Delta v_n|^2 dx = \int_{\mathbb{R}^N} |\Delta u_n|^2 dx$$

and

$$\int_\Omega \int_\Omega \frac{|v_n(x)|^{2_\alpha^*} |v_n(y)|^{2_\alpha^*} dxdy}{|x-y|^\alpha} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2_\alpha^*} |u_n(y)|^{2_\alpha^*} dxdy}{|x-y|^\alpha}.$$

This implies that $S_{H,L}(\Omega) \leq S_{H,L}$. As in Hardy-Littlewood-Sobolev inequality, the equality holds only with the help of family of minimizers $\widehat{U}_\epsilon(x)$, so $S_{H,L}(\Omega)$ is achieved only when $\Omega = \mathbb{R}^N$. □

Now, we recall Lemmas 2.2.2 and 2.2.3 which are used to prove the Lemma 2.2.4 for nonlocal term of the energy functional.

Lemma 2.2.2. *Let $\{u_n\}$ be a bounded sequence in $L^q(\mathbb{R}^N)$ for $q \in (1,\infty)$. If $u_n \to u$ a.e. in $\mathbb{R}^N$ as $n \to \infty$, then $u_n \rightharpoonup u$ weakly in $L^q(\mathbb{R}^N)$.*

Proof. Proof is followed by [69] (Prop. 5.4.7). □

Lemma 2.2.3. *Let $q \in [1,\infty)$ and $\{u_n\}$ be a bounded sequence in $L^r(\Omega)$. If $u_n \to u$ a.e. in Ω as $n \to \infty$, then for every $q \in [1,r]$*

$$|u_n|^q - |u_n - u|^q \to |u|^q \ \text{ in } \ L^{\frac{r}{q}}(\Omega) \ \text{ as } \ n \to \infty.$$

Proof. Proof is followed by [12] or [53] (Lemma 2.5). □

Lemma 2.2.4. *Let $N \geq 5$, $0 < \alpha < N$ and $\{u_n\}$ be a bounded sequence in $L^{\frac{2N}{N-4}}(\mathbb{R}^N)$. If $u_n \to u$ a.e. in $\mathbb{R}^N$ as $n \to \infty$, then*

$$\lim_{n\to\infty} \left(\int_{\mathbb{R}^N} (|x|^{-\alpha} * |u_n|^{2_\alpha^*}) |u_n|^{2_\alpha^*} - \int_{\mathbb{R}^N} (|x|^{-\alpha} * |u_n - u|^{2_\alpha^*}) |u_n - u|^{2_\alpha^*} \right)$$
$$= \int_{\mathbb{R}^N} (|x|^{-\alpha} * |u|^{2_\alpha^*}) |u|^{2_\alpha^*}.$$

Proof. The proof is similar to the proof of the Brézis-Lieb Lemma (see [12]) or Lemma 2.2 [32]. But for completeness, we give the detail. Consider

$$\begin{aligned}\int_{\mathbb{R}^N}(|x|^{-\alpha}*|u_n|^{2^*_\alpha})|u_n|^{2^*_\alpha}-\int_{\mathbb{R}^N}(|x|^{-\alpha}*|u_n-u|^{2^*_\alpha})|u_n-u|^{2^*_\alpha}\\ =\int_{\mathbb{R}^N}(|x|^{-\alpha}*(|u_n|^{2^*_\alpha}-|u_n-u|^{2^*_\alpha}))(|u_n|^{2^*_\alpha}-|u_n-u|^{2^*_\alpha})\\ +2\int_{\mathbb{R}^N}(|x|^{-\alpha}*(|u_n|^{2^*_\alpha}-|u_n-u|^{2^*_\alpha}))|u_n-u|^{2^*_\alpha}.\end{aligned} \tag{2.2.9}$$

Now use Lemma 2.2.3, for $q=2^*_\alpha=\frac{2N-\alpha}{N-4}$ and $r=\frac{2N}{2N-\alpha}2^*_\alpha$, then we obtain

$$|u_n|^{2^*_\alpha}-|u_n-u|^{2^*_\alpha}\to|u|^{2^*_\alpha}\ \text{ in }\ L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N)\ \text{ as }\ n\to\infty. \tag{2.2.10}$$

Also the Hardy-Littlewood-Sobolev inequality implies that

$$|x|^{-\alpha}*(|u_n|^{2^*_\alpha}-|u_n-u|^{2^*_\alpha})\to|x|^{-\alpha}*|u|^{2^*_\alpha}\ \text{ in }\ L^{\frac{2N}{\alpha}}(\mathbb{R}^N)\ \text{ as }\ n\to\infty. \tag{2.2.11}$$

Hence by Lemma 2.2.2, we obtain $|u_n-u|^{2^*_\alpha}\rightharpoonup 0$ weakly in $L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N)$ as $n\to\infty$. So using this together with (2.2.10), (2.2.11), in (2.2.9), we obtain the required result. □

Lemma 2.2.5. *Let $N\geq 5$, $0<\alpha<N$ and*

$$\|.\|_{NL}=\left(\int_\Omega\int_\Omega\frac{|.|^{2^*_\alpha}|.|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{22^*_\alpha}}$$

with $Y_{NL}=\{u:\Omega\to\mathbb{R};\|u\|_{NL}<\infty\}$. Then $\|.\|_{NL}$ defines a norm in Y_{NL} and Y_{NL} is a Banach space.

2.3 Nehari manifold and fibering map analysis

In this section, we examine the nature of Nehari manifold with the help of fibering maps of corresponding energy functional $\mathcal{I}_\lambda$. Since the energy functional $\mathcal{I}_\lambda$ is unbounded below on $H^2_0(\Omega)$. So we restrict $\mathcal{I}_\lambda$ to a suitable subset $\mathcal{N}_\lambda$ of $H^2_0(\Omega)$, called

Nehari set, defined as follows

$$\mathcal{N}_\lambda = \left\{ u \in H_0^2(\Omega) \setminus \{0\} : \langle \mathcal{I}'_\lambda(u), u \rangle = 0 \right\}.$$

We note that $u \in \mathcal{N}_\lambda$ if and only if

$$\langle \mathcal{I}'_\lambda(u), u \rangle = \|u\|^2 - \int_\Omega \int_\Omega \frac{g(x)g(y)|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy - \lambda \int_\Omega f(x)|u|^q dx = 0. \quad (2.3.12)$$

It means that $\mathcal{N}_\lambda$ contains every nontrivial solution of (E_λ). We see in the following Lemma that $\mathcal{I}_\lambda$ is bounded below and coercive.

Lemma 2.3.1. *The energy functional $\mathcal{I}_\lambda$ is coercive and bounded on $\mathcal{N}_\lambda$.*

Proof. Let $u \in \mathcal{N}_\lambda$, then using (2.3.12), Hölder's inequality, Sobolev embedding and (2.2.7), we have

$$\begin{aligned} \mathcal{I}_\lambda(u) &= \left(\frac{1}{2} - \frac{1}{2.2^*_\alpha}\right)\|u\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{2.2^*_\alpha}\right)\int_\Omega f(x)|u|^q dx \\ &\geq \left(\frac{1}{2} - \frac{1}{2.2^*_\alpha}\right)\|u\|^2 - \lambda\|f^+\|_\infty |\Omega|^{\frac{2^*-q}{2^*}} \left(\frac{1}{q} - \frac{1}{2.2^*_\alpha}\right)\|u\|^q_{2^*} \\ &\geq \left(\frac{1}{2} - \frac{1}{2.2^*_\alpha}\right)\|u\|^2 - \lambda\|f^+\|_\infty S^{-\frac{q}{2}} |\Omega|^{\frac{2^*-q}{2^*}} \left(\frac{1}{q} - \frac{1}{2.2^*_\alpha}\right)\|u\|^q \\ &= \left(\frac{1}{2} - \frac{1}{2.2^*_\alpha}\right)\|u\|^2 - \lambda\|f^+\|_\infty |\Omega|^{\frac{2^*-q}{2^*}} \frac{C(N,\alpha)^{\frac{-q}{2.2^*_\alpha}}}{S_{H,L}^{\frac{q}{2}}} \left(\frac{1}{q} - \frac{1}{2.2^*_\alpha}\right)\|u\|^q. \quad (2.3.13) \end{aligned}$$

Since $1 < q < 2$, we have, $\mathcal{I}_\lambda$ is coercive and bounded below on $\mathcal{N}_\lambda$. □

The Nehari manifold is closely related to the behavior of the mapping $\Phi_u : t \to \mathcal{I}_\lambda(tu)$. These maps are known as fibering maps and given by Drabek and Pohozaev [24] and further discussed by Brown and Zhang [16]. Thus, we have

$$\begin{aligned} \Phi_u(t) &= \mathcal{I}_\lambda(tu) = \frac{t^2}{2}\|u\|^2 - \frac{t^{2.2^*_\alpha}}{2.2^*_\alpha}\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha} - \frac{\lambda t^q}{q}\int_\Omega f(x)|u|^q dx. \\ \Phi'_u(t) &= t\|u\|^2 - t^{2.2^*_\alpha - 1}\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha} - \lambda t^{q-1}\int_\Omega f(x)|u|^q dx. \\ \Phi''_u(t) &= \|u\|^2 - (2.2^*_\alpha - 1)t^{2.2^*_\alpha - 2}\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha} - \lambda(q-1)t^{q-2}\int_\Omega f|u|^q. \end{aligned}$$

It is easy to see that $\Phi_u'(t) = 0$ if and only if $tu \in \mathcal{N}_\lambda$, i.e. $u \in \mathcal{N}_\lambda$ iff $\Phi_u'(1) = 0$. Therefore, it is natural to split $\mathcal{N}_\lambda$ into three parts $\mathcal{N}_\lambda^+, \mathcal{N}_\lambda^-$ and $\mathcal{N}_\lambda^0$ corresponding to local minima, local maxima and point of inflection respectively. Thus, we have

$$\mathcal{N}_\lambda^\pm := \{u \in \mathcal{N}_\lambda : \Phi_u''(1) \gtrless 0\} \text{ and } \mathcal{N}_\lambda^0 := \{u \in \mathcal{N}_\lambda : \Phi_u''(1) = 0\}.$$

Now, for $u \in \mathcal{N}_\lambda$, we have

$$\Phi_u''(1) = \begin{cases} (2 - 2.2_\alpha^*)\|u\|^2 - \lambda(q - 2.2_\alpha^*) \int_\Omega f(x)|u|^q dx \\ (2 - q)\|u\|^2 - (2.2_\alpha^* - q) \int_\Omega \int_\Omega g(x)g(y) \frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^\alpha} dxdy. \end{cases} \tag{2.3.14}$$

Lemma 2.3.2. *If u is the local minimizer for $\mathcal{I}_\lambda$ on subsets $\mathcal{N}_\lambda^+$ or $\mathcal{N}_\lambda^-$ of $\mathcal{N}_\lambda$ such that $u \notin \mathcal{N}_\lambda^0$. Then $\mathcal{I}_\lambda'(u) = 0$ in $(H_0^2(\Omega))^{-1}$, where $(H_0^2(\Omega))^{-1}$ denotes the dual space of $H_0^2(\Omega)$.*

Proof. The proof follows the same as done in Lemma 1.3.1 in chapter 1. □

Lemma 2.3.3. *We have the following:*
(i) If $u \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0$, then $\lambda \int_\Omega f(x)|u|^q dx > 0$.
(ii) If $u \in \mathcal{N}_\lambda^- \cup \mathcal{N}_\lambda^0$, then $\int_\Omega \int_\Omega g(x)g(y) \frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^\alpha} dxdy > 0$.

Proof. Proof directly follows from the equations (2.3.14), and the definition of $\mathcal{N}_\lambda^\pm$ and $\mathcal{N}_\lambda^0$. □

Now, for each $u \in H_0^2(\Omega)$, define a map $\xi_u : \mathbb{R}^+ \to \mathbb{R}$ by

$$\xi_u(t) = t^{2-q}\|u\|^2 - t^{2.2_\alpha^* - q} \int_\Omega \int_\Omega g(x)g(y) \frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^\alpha}.$$

Clearly, for $t > 0$, $tu \in \mathcal{N}_\lambda$ if and only if t is a solution of $\xi_u(t) = \lambda \int_\Omega f(x)|u|^q dx$. It is easy to see that

$$\xi_u'(t) = (2 - q)t^{1-q}\|u\|^2 - (2.2_\alpha^* - q)t^{2.2_\alpha^* - q - 1} \int_\Omega \int_\Omega g(x)g(y) \frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^\alpha},$$

and $\xi_u'(t) = 0$ if and only if $t = t_{\max}$, where

$$t_{\max} = \left(\frac{(2-q)\|u\|^2}{(2.2_\alpha^* - q)\int_\Omega \int_\Omega g(x)g(y)\frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^\alpha}} \right)^{\frac{1}{2.2_\alpha^*-2}}.$$

Moreover, if $\int_\Omega \int_\Omega g(x)g(y)\frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^\alpha} > 0$, then one can easily see that $\xi_u(0) = 0$, $\xi_u(t) \to -\infty$ as $t \to \infty$ and $\xi_u(t)$ attains its maximum value at $t_{\max}$.

Lemma 2.3.4. *Suppose $0 < \lambda < \Gamma_1$ and $u \in H_0^2(\Omega)$, then the following results hold:*

(i) If $\int_\Omega f(x)|u|^q dx < 0$ and $\displaystyle\int_\Omega \int_\Omega g(x)g(y)\frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^\alpha} < 0$, then there exists no critical point.

(ii) If $\int_\Omega f(x)|u|^q dx > 0$ and $\displaystyle\int_\Omega \int_\Omega g(x)g(y)\frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^\alpha} \leq 0$, then there exists a unique $t^+(u)$ such that $t^+u \in \mathcal{N}_\lambda^+$ and $\mathcal{I}_\lambda(t^+u) = \inf\limits_{t\geq 0} \mathcal{I}_\lambda(tu)$.

(iii) If $\displaystyle\int_\Omega \int_\Omega g(x)g(y)\frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^\alpha} > 0$ and $\int_\Omega f(x)|u|^q dx \leq 0$, then there exists a unique $t^-(u) > t_{\max}$ such that $t^-(u)u \in \mathcal{N}_\lambda^-$ and $\mathcal{I}_\lambda(t^-u) = \sup\limits_{t\geq t_{\max}} \mathcal{I}_\lambda(tu)$.

(iv) If $\displaystyle\int_\Omega \int_\Omega g(x)g(y)\frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^\alpha} > 0$ and $\int_\Omega f(x)|u|^q dx > 0$, then there exists unique t^+ and t^- satisfying $0 < t^+ < t_{\max} < t^-$ such that $t^+(u)u \in \mathcal{N}_\lambda^+$ and $t^-(u)u \in \mathcal{N}_\lambda^-$. Moreover

$$\mathcal{I}_\lambda(t^+u) = \inf_{0\leq t\leq t_{\max}} \mathcal{I}_\lambda(tu);\ \mathcal{I}_\lambda(t^-u) = \sup_{t\geq t_{\max}} \mathcal{I}_\lambda(tu).$$

Proof. If $0 \neq u \in H_0^2(\Omega)$ satisfying

(i) $\int_\Omega f(x)|u|^q dx < 0$ and $\displaystyle\int_\Omega \int_\Omega g(x)g(y)\frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^\alpha} < 0$, then $\Phi_u(0) = 0$, $\Phi_u'(t) > 0$ for all $t > 0$, which implies that Φ_u is increasing and hence no critical point.

(ii) If $\int_\Omega \int_\Omega g(x)g(y)\frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^\alpha} < 0$, then ξ_u is strictly increasing for $t > 0$ and $\xi_u(0) = 0$. As $\int_\Omega f(x)|u|^q dx \geq 0$, so there exists a unique $t^+ := t^+(u)$ such that $\xi_u(t^+(u)) = \lambda \int_\Omega f(x)|u|^q dx$ and $\xi_u(t^+) > 0$. Now, the relation $\Phi_u'(t) = t^q(\xi_u(t) - \lambda \int_\Omega f(x)|u|^q dx)$ implies that $\Phi_u'(t^+(u)) = 0$. Thus,

$t^+(u)u \in \mathcal{N}_\lambda$. Also $\Phi'_u(t) > 0$ for $t > t^+(u)$ and $\Phi'_u(t) < 0$ for $t < t^+(u)$ and $\Phi''_u(t^+(u)) = (t^+(u))^{q+1}\xi'_u(t^+(u)) > 0$. Hence $t^+(u)u \in \mathcal{N}_\lambda^+$ and $\mathcal{I}_\lambda(t^+(u)u) = \inf_{t\geq 0} \mathcal{I}_\lambda(tu)$.

(*iii*) If $\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha} > 0$, then $\xi'_u(t) > 0$ for $t \in [0, t_{\max})$ and $\xi'_u(t) < 0$ for $t \in (t_{\max}, \infty)$. Thus, $\xi_u(t)$ attains its maximum at $t_{\max}$. Since $\lambda\int_\Omega f(x)|u|^q dx \leq 0$, there is a unique $t^-(u) > t_{\max}(u) > 0$ such that $\xi_u(t^-(u)) = \lambda\int_\Omega f(x)|u|^q dx$ and $\xi_u(t^-) < 0$. Now, the relation $\Phi'_u(t) = t^q(\xi_u(t) - \lambda\int_\Omega f(x)|u|^q dx)$ implies that $\Phi'_u(t^-(u)) = 0$. Thus $t^-(u)u \in \mathcal{N}_\lambda$. Moreover, $\Phi''_u(t^-(u)) = (t^-(u))^{q+1}\xi'_u(t^-(u)) < 0$ implies $t^-(u)u \in \mathcal{N}_\lambda^-$. Also $\Phi'_u(t) < 0$ for $t > t_{\max}$, so $\Phi_u(t^-) = \sup_{t\geq t_{\max}} \Phi_u(t)$.
Hence $\mathcal{I}_\lambda(t^- u) = \sup_{t\geq t_{\max}} \mathcal{I}_\lambda(tu)$.

(*iv*) Since $\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha} > 0$, ξ_u attains its maximum value at $t = t_{\max}$ and

$$\begin{aligned}\xi_u(t_{\max}) =& \|u\|^q \left(\frac{2-q}{2.2^*_\alpha - q}\right)^{\frac{2-q}{2.2^*_\alpha-2}} \left(\frac{2.2^*_\alpha - 2}{2.2^*_\alpha - q}\right) \left(\frac{\|u\|^{2.2^*_\alpha}}{\int_\Omega\int_\Omega \frac{g(x)g(y)|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}}\right)^{\frac{2-q}{2.2^*_\alpha-2}} \\ \geq& \left[\left(\frac{2-q}{2.2^*_\alpha - q}\right)\|g^+\|_\infty^{-2} S_{H,L}^{2^*_\alpha}\right]^{\frac{2-q}{2.2^*_\alpha-2}} \left(\frac{2.2^*_\alpha - 2}{2.2^*_\alpha - q}\right)\|u\|^q.\end{aligned}$$

Now, if $\int_\Omega f(x)|u|^q dx > 0$, then

$$\begin{aligned}\xi_u(t_{\max}) - \lambda\int_\Omega f|u|^q \geq& \left(\frac{2.2^*_\alpha - 2}{2.2^*_\alpha - q}\right)\left[\left(\frac{2-q}{2.2^*_\alpha - q}\right)\|g^+\|_\infty^{-2} S_{H,L}^{2^*_\alpha}\right]^{\frac{2-q}{2.2^*_\alpha-2}} \|u\|^q \\ &- \lambda\|f^+\|_\infty|\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}}\|u\|^q > 0,\end{aligned}$$

for $0 < \lambda < \Gamma_1$, where Γ_1 is same as in (2.1.1). Thus, there exists $t^+(u)$ and

$t^-(u)$ such that $0 < t^+(u) < t_{\max} < t^-(u)$,

$$\xi_u(t^+(u)) = \lambda \int_\Omega f(x)|u|^q dx = \xi_u(t^-(u)) \text{ and } \xi_u'(t^-(u)) < 0 < \xi_u'(t^+(u)).$$

Therefore, $t^+(u)u \in \mathcal{N}_\lambda^+$, $t^-(u)u \in \mathcal{N}_\lambda^-$. Moreover, $\Phi_u'(t) < 0$ if $t \in (0, t^+(u))$, $\Phi_u'(t) > 0$ for $t \in (t^+, t^-)$ and $\Phi_u'(t) < 0$ for $t \in (t^-(u), \infty)$. Hence, we have

$$\mathcal{I}_\lambda(t^+(u)u) = \inf_{0 \le t \le t_{\max}} \mathcal{I}_\lambda(tu);\ \mathcal{I}_\lambda(t^-(u)u) = \sup_{t \ge t_{\max}} \mathcal{I}_\lambda(tu).$$

This completes the proof. □

Lemma 2.3.5. *If $0 < \lambda < \Gamma_1$, then $\mathcal{N}_\lambda^0 = \phi$, where Γ_1 is defined by* (2.1.1).

Proof. On contrary, assume that there exists $\lambda \in \mathbb{R}^+$ with $0 < \lambda < \Gamma_1$ such that $\mathcal{N}_\lambda^0 \neq \phi$. Then for $u \in \mathcal{N}_\lambda^0$, using (2.3.14), we obtain

$$\|u\|^2 = \lambda \frac{2.2_\alpha^* - q}{2.2_\alpha^* - 2} \int_\Omega f(x)|u|^q dx \tag{2.3.15}$$

and

$$\|u\|^2 = \frac{2.2_\alpha^* - q}{2 - q} \int_\Omega \int_\Omega g(x)g(y) \frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^\alpha} dxdy. \tag{2.3.16}$$

Now, using Hölder's inequality and Sobolev embedding theorem in (2.3.15), we obtain

$$\|u\|^2 \le \lambda \frac{2.2_\alpha^* - q}{2.2_\alpha^* - 2} \|f^+\|_\infty |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \|u\|^q$$

and so

$$\|u\| \le \left(\lambda \frac{2.2_\alpha^* - q}{2.2_\alpha^* - 2} \|f^+\|_\infty |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \right)^{\frac{1}{2-q}}. \tag{2.3.17}$$

Again from (2.3.16), (2.2.4), Hölder's inequality and (2.2.7), we have

$$
\begin{aligned}
\|u\|^2 &\leq \frac{2.2^*_\alpha - q}{2-q}\|g^+\|^2_\infty C(N,\alpha)S^{-2^*_\alpha}\|u\|^{2.2^*_\alpha} \\
&\leq \frac{2.2^*_\alpha - q}{2-q}\|g^+\|^2_\infty S_{H,L}^{-2^*_\alpha}\|u\|^{2.2^*_\alpha},
\end{aligned}
$$

which implies that

$$
\|u\| \geq \left(\frac{2-q}{2.2^*_\alpha - q}\|g^+\|^{-2}_\infty S_{H,L}^{2^*_\alpha}\right)^{\frac{1}{2.2^*_\alpha - 2}}. \tag{2.3.18}
$$

Thus, from (2.3.17) and (2.3.18), we obtain

$$
\lambda \geq \left(\frac{2-q}{2.2^*_\alpha - q}\|g^+\|^{-2}_\infty S_{H,L}^{2^*_\alpha}\right)^{\frac{2-q}{2.2^*_\alpha - 2}} \left(\frac{2.2^*_\alpha - 2}{2.2^*_\alpha - q}\right)\|f^+\|^{-1}_\infty |\Omega|^{\frac{q-2^*}{2^*}} S^{\frac{q}{2}} := \Gamma_1,
$$

which contradicts the fact that $0 < \lambda < \Gamma_1$. Hence $\mathcal{N}_\lambda^0 = \phi$. □

Thus, we can write, if $0 < \lambda < \Gamma_1$, then we have $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$. Now, we define

$$
\theta_\lambda = \inf_{u \in \mathcal{N}_\lambda} \mathcal{I}_\lambda(u) \;\; ; \;\; \theta_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} \mathcal{I}_\lambda(u) \;\; ; \;\; \theta_\lambda^- = \inf_{u \in \mathcal{N}_\lambda^-} \mathcal{I}_\lambda(u).
$$

Now, we have the following results:

Lemma 2.3.6. *The following facts hold:*

(*i*) *If* $0 < \lambda < \Gamma_1$, *then* $\theta_\lambda \leq \theta_\lambda^+ < 0$.

(*ii*) *If* $0 < \lambda < \frac{q}{2}\Gamma_1$, *then* $\theta_\lambda^- > C_0$, *where* C_0 *is a positive constant depending on* λ, q, N, $C(N,\alpha)$, $|\Omega|, S_{H,L}, \|f^+\|_\infty$ *and* $\|g^+\|_\infty$.

Proof. (*i*) Assume $u \in \mathcal{N}_\lambda^+$. Then by (2.3.14), we have

$$
\frac{2-q}{2.2^*_\alpha - q}\|u\|^2 > \int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy. \tag{2.3.19}
$$

Using (2.1.2), (2.3.12) and (2.3.19),

$$\begin{aligned}\mathcal{I}_\lambda(u) &= \left(\frac{1}{2}-\frac{1}{q}\right)\|u\|^2+\left(\frac{1}{q}-\frac{1}{2.2^*_\alpha}\right)\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\\ &< -\frac{(2-q)(2^*_\alpha-1)}{2.2^*_\alpha q}\|u\|^2<0.\end{aligned}$$

Thus, by the definition of θ_λ and θ_λ^+, we conclude that $\theta_\lambda \le \theta_\lambda^+ < 0$.

(ii) Let $u\in\mathcal{N}_\lambda^-$. Then using (2.3.14) and (2.2.4), we have

$$\frac{2-q}{2.2^*_\alpha-q}\|u\|^2<\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\le\|g^+\|^2_\infty S_{H,L}^{-2^*_\alpha}\|u\|^{2.2^*_\alpha},$$

which implies that

$$\|u\|>\left(\frac{2-q}{2.2^*_\alpha-q}\|g^+\|_\infty^{-2}S_{H,L}^{2^*_\alpha}\right)^{\frac{1}{2.2^*_\alpha-2}}. \tag{2.3.20}$$

Now, equations (2.3.13) and (2.3.20) give

$$\begin{aligned}\mathcal{I}_\lambda(u) \ge&\|u\|^q\left[\left(\frac{1}{2}-\frac{1}{2.2^*_\alpha}\right)\|u\|^{2-q}-\lambda\|f^+\|_\infty|\Omega|^{\frac{2^*-q}{2^*}}S_{H,L}^{-\frac{q}{2}}(C(N,\alpha))^{-\frac{q}{2.2^*_\alpha}}\left(\frac{1}{q}-\frac{1}{2.2^*_\alpha}\right)\right]\\ >&\left[\frac{2^*_\alpha-1}{2.2^*_\alpha}\left(\frac{2-q}{2.2^*_\alpha-q}\|g^+\|_\infty^{-2}S_{H,L}^{2^*_\alpha}\right)^{\frac{2-q}{2.2^*_\alpha-q}}-\frac{2.2^*_\alpha-q}{2.2^*_\alpha q}\lambda\|f^+\|_\infty|\Omega|^{\frac{2^*-q}{2^*}}\frac{C(N,\alpha)^{-\frac{q}{2.2^*_\alpha}}}{S_{H,L}^{\frac{q}{2}}}\right]\\ &\times\left(\frac{2-q}{2.2^*_\alpha-q}\|g^+\|_\infty^{-2}S_{H,L}^{2^*_\alpha}\right)^{\frac{q}{2.2^*_\alpha-2}}.\end{aligned}$$

Thus, if $0<\lambda<\frac{q}{2}\Gamma_1$, then $\mathcal{I}_\lambda(u)>C_0$ for all $u\in\mathcal{N}_\lambda^-$, where C_0 is a positive constant depending on λ, q, N, $C(N,\alpha)$, $|\Omega|$, $S_{H,L}$, $\|f^+\|_\infty$ and $\|g^+\|_\infty$. □

Corollary 2.3.1. $\mathcal{N}_\lambda^-$ *is a closed set.*

Proof. Let $\{u_n\}$ be a sequence in $\mathcal{N}_\lambda^-$ such that $u_n\to u$ in $H_0^2(\Omega)$. Our aim is to show that $u\in\mathcal{N}_\lambda^-$. Since $\{u_n\}\subseteq\mathcal{N}_\lambda^-$, we get

$$\Phi''_{u_n}(1)=(2-2.2^*_\alpha)\|u_n\|^2-\lambda(q-2.2^*_\alpha)\int_\Omega f(x)|u_n|^q dx<0.$$

On passing the limit, we have

$$\Phi_u''(1) = (2 - 2.2_\alpha^*)\|u\|^2 - \lambda(q - 2.2_\alpha^*)\int_\Omega f(x)|u|^q dx \leq 0.$$

If $\Phi_u''(1) = 0$, then $\mathcal{N}_\lambda^0 \neq \phi$, which is a contradiction to Lemma 2.3.5. Thus $\Phi_u''(1) < 0$ and so $u \in \mathcal{N}_\lambda^-$. □

Lemma 2.3.7. *Suppose $0 < \lambda < \Gamma_1$, where Γ_1 is same as defined in (2.1.1). Then for every $u \in \mathcal{N}_\lambda$, there exists $\epsilon > 0$ and a differentiable function $\zeta : B(0,\epsilon) \subset H_0^2(\Omega) \to \mathbb{R}^+$ such that $\zeta(0) = 1$ and $\zeta(w)(u-w) \in \mathcal{N}_\lambda$ and for all $w \in H_0^2(\Omega)$*

$$\langle \zeta'(0), w\rangle = \frac{2\int_\Omega \Delta u \Delta w dx - G(u,w) - q\lambda \int_\Omega f(x)|u|^{q-2}uw dx}{(2-q)\|u\|^2 - (2.2_\alpha^* - q)\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^\alpha}}, \tag{2.3.21}$$

where $G(u,w) = 2.2_\alpha^* \int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2_\alpha^*-2}u(x)|u(y)|^{2_\alpha^*}w(x)}{|x-y|^\alpha}dxdy$.

Proof. For $u \in \mathcal{N}_\lambda$, define a function $K_u : \mathbb{R} \times H_0^2(\Omega) \to \mathbb{R}$ such that

$$\begin{aligned} K_u(t,w) &= \langle \mathcal{I}_\lambda'(t(u-w)), t(u-w)\rangle \\ &= t^2\|u-w\|^2 - t^q\lambda\int_\Omega f(x)|u-w|^q \\ &\qquad - t^{2.2_\alpha^*}\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)-w(x)|^{2_\alpha^*}|u(y)-w(y)|^{2_\alpha^*}}{|x-y|^\alpha}. \end{aligned}$$

Then $K_u(1,0) = \langle \mathcal{I}_\lambda'(u), u\rangle$ and

$$\begin{aligned} \frac{d}{dt}K_u(1,0) &= 2\|u\|^2 - \lambda q\int_\Omega f(x)|u(x)|^q dx - 2.2_\alpha^*\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^\alpha} \\ &= (2-q)\|u\|^2 - (2.2_\alpha^* - q)\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^\alpha} \neq 0. \end{aligned}$$

By implicit function theorem, there exists $\epsilon > 0$ and a differentiable function $\zeta : B(0,\epsilon) \subset H_0^2(\Omega) \to \mathbb{R}$ such that $\zeta(0) = 1$,

$$\langle \zeta'(0), w\rangle = \frac{2\int_\Omega \Delta u \Delta w dx - G(u,w) - q\lambda \int_\Omega f(x)|u|^{q-2}uw dx}{(2-q)\|u\|^2 - (2.2_\alpha^* - q)\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^\alpha}}$$

and $K_u(\zeta(w),w)=0$ for all $w\in B(0,\epsilon)$ which means that $\langle \mathcal{I}'_\lambda(\zeta(w)(u-w)),\zeta(w)(u-w)\rangle=0$ for all $u\in B(0,\epsilon)$. Hence $\zeta(w)(u-w)\in\mathcal{N}_\lambda$. $\square$

Lemma 2.3.8. *Suppose* $0<\lambda<\Gamma_1$, *where* Γ_1 *is same as defined in* (2.1.1)*. Then for every* $u\in\mathcal{N}_\lambda^-$, *there exists* $\epsilon>0$ *and a differentiable function* $\zeta^-:B(0,\epsilon)\subset H_0^2(\Omega)\to\mathbb{R}^+$ *such that* $\zeta^-(0)=1$ *and* $\zeta^-(w)(u-w)\in\mathcal{N}_\lambda^-$ *and for all* $w\in H_0^2(\Omega)$

$$\langle(\zeta^-)'(0),w\rangle=\frac{2\int_\Omega \Delta u\Delta w dx-G(u,w)-q\lambda\int_\Omega f(x)|u|^{q-2}uwdx}{(2-q)\|u\|^2-(2.2^*_\alpha-q)\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}},$$

where $G(u,w)$ *is same as given in Lemma 2.3.7.*

Proof. Similar to the argument in Lemma 2.3.7, there exists $\epsilon>0$ and a differentiable function $\zeta^-:B(0,\epsilon)\subset H_0^2(\Omega)\to\mathbb{R}^+$ such that $\zeta^-(0)=1$ and $\zeta^-(w)(u-w)\in\mathcal{N}_\lambda^-$. Since

$$\Phi''_u(1)=(2-q)\|u\|^2-(2.2^*_\alpha-q)\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}<0.$$

Thus, by continuity of the functions Φ'', ζ^-, we have

$$\begin{aligned}\Phi''_{\zeta^-(w)(u-w)}(1)&=(2-q)\|\zeta^-(w)(u-w)\|^2\\&\quad-(2.2^*_\alpha-q)\int_\Omega\int_\Omega g(x)g(y)\frac{|\zeta^-(w)(u-w)|^{2^*_\alpha}|\zeta^-(w)(u-w)|^{2^*_\alpha}}{|x-y|^\alpha}<0,\end{aligned}$$

if ϵ is sufficiently small, which implies $\zeta^-(w)(u-w)\in\mathcal{N}_\lambda^-$. $\square$

Lemma 2.3.9. *The following hold:*

(*i*) *If* $0<\lambda<\Gamma_1$, *then there exists a* $(PS)_{\theta_\lambda}$*-sequence* $\{u_n\}\subset\mathcal{N}_\lambda$ *in* $H_0^2(\Omega)$ *for* $\mathcal{I}_\lambda$.

(*ii*) *If* $0<\lambda<\frac{q}{2}\Gamma_1$, *then there exists a* $(PS)_{\theta_\lambda^-}$*-sequence* $\{u_n\}\subset\mathcal{N}_\lambda^-$ *in* $H_0^2(\Omega)$ *for* $\mathcal{I}_\lambda$,

where Γ_1 *is same as given in* (2.1.1).

Proof. By Lemma 2.3.1 and Ekeland variational principle [27], there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda$ such that

$$\mathcal{I}_\lambda(u_n) < \theta_\lambda + \frac{1}{n}, \tag{2.3.22}$$
$$\mathcal{I}_\lambda(u_n) < \mathcal{I}_\lambda(u) + \frac{1}{n}\|u - u_n\|, \text{ for each } u \in \mathcal{N}_\lambda.$$

Taking n large and using (2.3.22) together with Lemma 2.3.6 (i), we obtain

$$\mathcal{I}_\lambda(u_n) = \left(\frac{1}{2} - \frac{1}{2.2^*_\alpha}\right)\|u_n\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{2.2^*_\alpha}\right)\int_\Omega f(x)|u_n|^q dx < \ \theta_\lambda + \frac{1}{n} < \frac{\theta_\lambda}{2}. \tag{2.3.23}$$

This implies

$$0 < -\frac{2.2^*_\alpha q\theta_\lambda}{2(2.2^*_\alpha - q)} < \int_\Omega f(x)|u_n|^q dx \leq \|f^+\|_\infty S^{-\frac{q}{2}}|\Omega|^{\frac{2^*-q}{2^*}}\|u_n\|^q. \tag{2.3.24}$$

Consequently $u_n \neq 0$. Now from (2.3.23) and Hölder's inequality, we have

$$\|u_n\| < \left(\frac{2\lambda(2.2^*_\alpha - q)}{q(2.2^*_\alpha - 2)}\|f^+\|_\infty|\Omega|^{\frac{2^*-q}{2^*}}S^{-\frac{q}{2}}\right)^{\frac{1}{2-q}} \tag{2.3.25}$$

and equation (2.3.24) gives

$$\|u_n\| > \left(\frac{-2.2^*_\alpha q}{2(2.2^*_\alpha - q)}\theta_\lambda\|f^+\|_\infty^{-1}|\Omega|^{\frac{q-2^*}{2^*}}S^{\frac{q}{2}}\right)^{\frac{1}{q}}. \tag{2.3.26}$$

Now, we will prove that $\|\mathcal{I}'_\lambda(u_n)\|_{(H^2_0(\Omega))^{-1}} \to 0$ as $n \to \infty$.
Using Lemma 2.3.7 with u_n to obtain a sequence of functions $\zeta_n : B(0, \epsilon_n) \to \mathbb{R}^+$ for some $\epsilon_n > 0$ such that $\zeta_n(w)(u_n - w) \in \mathcal{N}_\lambda$. Choose $0 < \eta < \epsilon_n$. Let $u \in H^2_0(\Omega)$ with $u \neq 0$ then take $w^*_\eta = \frac{\eta u}{\|u\|}$ and set $w_\eta = \zeta_n(w^*_\eta)(u_n - w^*_\eta)$. Since $w_\eta \in \mathcal{N}_\lambda$, from (2.3.22), we obtain

$$\mathcal{I}_\lambda(w_\eta) - \mathcal{I}_\lambda(u_n) \geq -\frac{1}{n}\|w_\eta - u_n\|.$$

By mean value theorem, we have

$$\langle \mathcal{I}'_\lambda(u_n), w_\eta - u_n\rangle + o(\|w_\eta - u_n\|) \geq -\frac{1}{n}\|w_\eta - u_n\|.$$

Hence

$$\langle \mathcal{I}'_\lambda(u_n), -w^*_\eta\rangle + \left(\zeta_n(w^*_\eta) - 1\right)\langle \mathcal{I}'_\lambda(u_n), (u_n - w^*_\eta)\rangle \geq -\frac{1}{n}\|w_\eta - u_n\| + o(\|w_\eta - u_n\|). \tag{2.3.27}$$

Since $\zeta_n(w^*_\eta)(u_n - w^*_\eta) \in \mathcal{N}_\lambda$ and from (2.3.27), we have

$$-\eta\left\langle \mathcal{I}'_\lambda(u_n), \frac{u}{\|u\|}\right\rangle + (\zeta_n(w^*_\eta) - 1)\langle \mathcal{I}'_\lambda(u_n) - \mathcal{I}'_\lambda(w_\eta), u_n - w^*_\eta\rangle \geq -\frac{1}{n}\|w_\eta - u_n\| \\ + o(\|w_\eta - u_n\|).$$

Thus,

$$\left\langle \mathcal{I}'_\lambda(u_n), \frac{u}{\|u\|}\right\rangle \leq \frac{1}{n\eta}\|w_\eta - u_n\| + \frac{1}{\eta}o(\|w_\eta - u_n\|) \\ + \frac{(\zeta_n(w^*_\eta) - 1)}{\eta}\langle \mathcal{I}'_\lambda(u_n - w_\eta), u_n - w^*_\eta\rangle. \tag{2.3.28}$$

Since $\|w_\eta - u_n\| \leq \eta|\zeta_n(w^*_\eta)| + |\zeta_n(w^*_\eta) - 1|\|u_n\|$ and $\lim_{\eta\to 0}\frac{|\zeta_n(w^*_\eta)-1|}{\eta} \leq \|\zeta'_n(0)\|$, if we take $\eta \to 0$ in (2.3.28) for a fixed $n \in \mathbb{N}$ and using (2.3.25), we can find a constant $M > 0$, independent from η such that

$$\left\langle \mathcal{I}'_\lambda(u_n), \frac{u}{\|u\|}\right\rangle \leq \frac{M}{n}(1 + \|\zeta'_n(0)\|).$$

Next, we show that $\|\zeta'_n(0)\|$ is uniformly bounded. From (2.3.21), (2.3.28), Hölder inequality and Lemma2.3.7, we have

$$\langle \zeta'_n(0), w\rangle \leq \frac{M_1\|w\|}{\left|(2-q)\|u_n\|^2 - (2.2^*_\alpha - q)\int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^{2^*_\alpha}|u_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right|},$$

for some $M_1 > 0$. Here we have used the Hardy-Littlewood-Sobolev inequality, Sobolev embedding and the fact that $\{u_n\}$ is bounded in $H_0^2(\Omega)$. Now, to complete the Lemma it is enough to show that

$$\left|(2-q)\|u_n\|^2 - (2.2_\alpha^* - q)\int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^{2_\alpha^*}|u_n(y)|^{2_\alpha^*}}{|x-y|^\alpha}dxdy\right| > d,$$

for some $d > 0$ and n large enough. We argue this by contradiction. Suppose that there exists a subsequence of $\{u_n\}$ still denoted as $\{u_n\}$ such that

$$(2-q)\|u_n\|^2 - (2.2_\alpha^* - q)\int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^{2_\alpha^*}|u_n(y)|^{2_\alpha^*}}{|x-y|^\alpha}dxdy = o_n(1). \quad (2.3.29)$$

Combining (2.3.26) and (2.3.29), we can find that there exists a constant $s > 0$ such that

$$\int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^{2_\alpha^*}|u_n(y)|^{2_\alpha^*}}{|x-y|^\alpha}dxdy \geq s,$$

for n sufficiently large. Now, using the fact that $u_n \in \mathcal{N}_\lambda$ together with (2.3.25) and (2.3.29), we have

$$\begin{aligned}\lambda\int_\Omega f(x)|u_n|^q dx &= \|u_n\|^2 - \int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^{2_\alpha^*}|u_n(y)|^{2_\alpha^*}}{|x-y|^\alpha} \\ &= \frac{2.2_\alpha^* - 2}{2-q}\int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^{2_\alpha^*}|u_n(y)|^{2_\alpha^*}}{|x-y|^\alpha} + o_n(1). \quad (2.3.30)\end{aligned}$$

Define $Q_\lambda : \mathcal{N}_\lambda \to \mathbb{R}$ such that

$$Q_\lambda(u) = C\left(\frac{\|u\|^{2(2.2_\alpha^*-1)}}{\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^\alpha}dxdy}\right)^{\frac{1}{2.2_\alpha^*-2}} - \lambda\int_\Omega f(x)|u|^q dx, \quad (2.3.31)$$

where $C = \left(\frac{2.2_\alpha^*-2}{2-q}\right)\left(\frac{2-q}{2.2_\alpha^*-q}\right)^{\frac{2.2_\alpha^*-1}{2.2_\alpha^*-2}}$. As $u_n \in \mathcal{N}_\lambda$ so using (2.3.30) and (2.3.29) in (2.3.31), we obtain

$$
\begin{aligned}
Q_\lambda(u_n) &= C\left[\frac{\left(\frac{2.2^*_\alpha-q}{2-q}\right)^{2.2^*_\alpha-1}\left(\int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^{2^*_\alpha}|u_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}\right)^{2.2^*_\alpha-1}}{\int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^{2^*_\alpha}|u_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy}\right]^{\frac{1}{2.2^*_\alpha-2}} \\
&\quad - \frac{2.2^*_\alpha-2}{2-q}\int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^{2^*_\alpha}|u_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy + o_n(1) \\
&= \left(\frac{2.2^*_\alpha-2}{2-q}\right)\left(\frac{2-q}{2.2^*_\alpha-q}\right)^{\frac{2.2^*_\alpha-1}{2.2^*_\alpha-2}}\times \\
&\quad \left(\frac{2.2^*_\alpha-q}{2-q}\right)^{\frac{2.2^*_\alpha-1}{2.2^*_\alpha-2}}\int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^{2^*_\alpha}|u_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy \\
&\quad - \frac{2.2^*_\alpha-2}{2-q}\int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^{2^*_\alpha}|u_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy + o_n(1) \\
&= o_n(1). \qquad (2.3.32)
\end{aligned}
$$

However, from (2.2.3), (2.3.17) and $\lambda \in (0, \Gamma_1)$, we have

$$
\begin{aligned}
Q_\lambda(u_n) &\geq C\left(\frac{\|u_n\|^{2(2.2^*_\alpha-1)}}{\|u_n\|^{2.2^*_\alpha}}\|g^+\|^2_\infty S^{2^*_\alpha}_{H,L}\right)^{\frac{1}{2.2^*_\alpha-2}} - \lambda\|f^+\|_\infty|\Omega|^{\frac{2^*-q}{2^*}}S^{-\frac{q}{2}}\|u_n\|^q \\
&= \|u_n\|^q\left(C\|u_n\|^{1-q}\|g^+\|_\infty^{-\frac{2}{2.2^*_\alpha-2}}S_{H,L}^{\frac{2^*_\alpha}{2.2^*_\alpha-2}} - \lambda\|f^+\|_\infty|\Omega|^{\frac{2^*-q}{2^*}}S^{-\frac{q}{2}}\right) \\
&\geq \|u_n\|^q\left[\left(\frac{2.2^*_\alpha-2}{2-q}\right)\left(\frac{2-q}{2.2^*_\alpha-q}\right)^{\frac{2.2^*_\alpha-1}{2.2^*_\alpha-2}}\|g^+\|_\infty^{-\frac{2}{2.2^*_\alpha-2}}S_{H,L}^{\frac{2^*_\alpha}{2.2^*_\alpha-2}}\right. \\
&\quad \left.\left(\lambda\frac{2.2^*_\alpha-q}{2.2^*_\alpha-2}\|f^+\|_\infty|\Omega|^{\frac{2^*-q}{2^*}}S^{-\frac{q}{2}}\right)^{\frac{1-q}{2-q}} - \lambda\|f^+\|_\infty|\Omega|^{\frac{2^*-q}{2^*}}S^{-\frac{q}{2}}\right] \\
&= \|u_n\|^q\left(\lambda^{\frac{1-q}{2-q}}\|f^+\|_\infty|\Omega|^{\frac{2^*-q}{2^*}}S^{-\frac{q}{2}}\Gamma_1^{\frac{1}{2-q}} - \lambda\|f^+\|_\infty|\Omega|^{\frac{2^*-q}{2^*}}S^{-\frac{q}{2}}\right) \\
&= \|u_n\|^q\lambda^{\frac{1-q}{2-q}}\|f^+\|_\infty|\Omega|^{\frac{2^*-q}{2^*}}S^{-\frac{q}{2}}\left(\Gamma_1^{\frac{1}{2-q}} - \lambda^{\frac{1}{2-q}}\right) > 0,
\end{aligned}
$$

which gives to (2.3.32). Thus, we obtain $\left\langle \mathcal{I}'_\lambda(u_n), \frac{u}{\|u\|}\right\rangle \leq \frac{c}{n}$. This completes the proof.

(ii) Using Lemma 2.3.8, proof follows similarly as given in (i) . □

2.4 The Palais-Smale condition

Lemma 2.4.1. *Assume that $(f1)$ and $(g1)$ hold. Suppose $\{u_n\} \subset H_0^2(\Omega)$ is a $(PS)_c$-sequence for $\mathcal{I}_\lambda$ such that $u_n \rightharpoonup u$ weakly in $H_0^2(\Omega)$. Then $\mathcal{I}'_\lambda(u) = 0$ and there exists a positive constant P_0 depending on N, α, q, S and $\|f^+\|_\infty$ such that*

$$\mathcal{I}_\lambda(u) \geq -P_0 \lambda^{\frac{2}{2-q}}.$$

Proof. If $\{u_n\}$ is a $(PS)_c$-sequence for $\mathcal{I}_\lambda$ with $u_n \rightharpoonup u$ weakly in $H_0^2(\Omega)$. Then using the standard argument, we obtain $\mathcal{I}'_\lambda(u) = 0$. Which implies $\langle \mathcal{I}'_\lambda(u), u \rangle = 0$, i.e,

$$\|u\|^2 - \int_\Omega \int_\Omega g(x)g(y) \frac{|u(x)|^{2^*_\alpha} |u(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy - \lambda \int_\Omega f(x)|u|^q dx = 0.$$

Consequently, using Hölder's inequality, Sobolev embedding theorem and Young's inequality, we have

$$\begin{aligned}
\mathcal{I}_\lambda(u) &= \frac{N+4-\alpha}{2(2N-\alpha)} \|u\|^2 - \lambda \frac{2.2^*_\alpha - q}{2.2^*_\alpha q} \int_\Omega f(x)|u|^q dx \\
&\geq \frac{N+4-\alpha}{2(2N-\alpha)} \|u\|^2 - \lambda \left(\frac{2.2^*_\alpha - q}{2.2^*_\alpha q} \right) \|f^+\|_\infty |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \|u\|^q \\
&\geq \frac{N+4-\alpha}{2(2N-\alpha)} \|u\|^2 - \left(\frac{2.2^*_\alpha - q}{2.2^*_\alpha q} \right) S^{-\frac{q}{2}} \left[\omega^{\frac{2}{2-q}} \left(\frac{2-q}{2} \right) \left(\lambda \|f^+\|_\infty |\Omega|^{\frac{2^*-q}{2^*}} \right)^{\frac{2}{2-q}} \right. \\
&\qquad \left. + \frac{q}{2} \omega^{-\frac{2}{q}} \|u\|^2 \right] \\
&= -P_0 \lambda^{\frac{2}{2-q}},
\end{aligned}$$

where $P_0 = \left(\frac{2.2^*_\alpha - q}{2.2^*_\alpha q} \right) S^{-\frac{q}{2}} \omega^{\frac{2}{2-q}} \left(\frac{2-q}{2} \right) \left(\|f^+\|_\infty |\Omega|^{\frac{2^*-q}{2^*}} \right)^{\frac{2}{2-q}}$, $\omega = \left[\frac{(2.2^*_\alpha - q)(2N-\alpha)}{2.2^*_\alpha (N+4-\alpha)} S^{-\frac{q}{2}} \right]^{\frac{q}{2}}$. This completes the proof. □

Lemma 2.4.2. *Suppose $\{u_n\} \subset H_0^2(\Omega)$ is a $(PS)_c$-sequence for $\mathcal{I}_\lambda$, then $\{u_n\}$ is bounded in $H_0^2(\Omega)$.*

Proof. Let $\{u_n\} \subset H_0^2(\Omega)$ be a $(PS)_c$-sequence for $\mathcal{I}_\lambda$, then by the definition of

$(PS)_c$-sequence, $\mathcal{I}_\lambda(u_n) \to c$ and $\mathcal{I}'_\lambda(u_n) \to 0$ in $(H_0^2(\Omega))^{-1}$. Thus, we have

$$\frac{\|u_n\|^2}{2} - \frac{1}{2.2^*_\alpha}\int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^{2^*_\alpha}|u_n(y)|^{2^*_\alpha}}{|x-y|^\alpha} - \frac{\lambda}{q}\int_\Omega f|u_n|^q = c + o_n(1). \quad (2.4.33)$$

$$\|u_n\|^2 - \int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^{2^*_\alpha}|u_n(y)|^{2^*_\alpha}}{|x-y|^\alpha} - \lambda\int_\Omega f|u_n|^q = o_n(1). \quad (2.4.34)$$

Our aim is to show that $\{u_n\}$ is bounded. We will prove it by contradiction. Suppose $\|u_n\| \to \infty$, then define $\overline{u}_n = \frac{u_n}{\|u_n\|}$ with $\|\overline{u}_n\| = 1$. This implies $\{\overline{u}_n\}$ is a bounded sequence. So, up to subsequence, $u_n \rightharpoonup \overline{u}$ weakly in $H_0^2(\Omega)$, $\overline{u}_n \to \overline{u}$ strongly in $L^m(\Omega)$, where $1 \le m < 2^*$, and $\overline{u}_n(x) \to \overline{u}(x)$ pointwise a.e. in Ω. Using (2.4.33) and (2.4.34), we obtain

$$\frac{\|\overline{u}_n\|^2}{2} - \frac{\|u_n\|^{2.2^*_\alpha-2}}{2.2^*_\alpha}\int_\Omega\int_\Omega \frac{g(x)g(y)|\overline{u}_n(x)|^{2^*_\alpha}|\overline{u}_n(y)|^{2^*_\alpha}}{|x-y|^\alpha} - \frac{\lambda\|u_n\|^{q-2}}{q}\int_\Omega f|\overline{u}_n|^q = o_n(1). \quad (2.4.35)$$

$$\|\overline{u}_n\|^2 - \|u_n\|^{2.2^*_\alpha-2}\int_\Omega\int_\Omega \frac{g(x)g(y)|\overline{u}_n(x)|^{2^*_\alpha}|\overline{u}_n(y)|^{2^*_\alpha}}{|x-y|^\alpha} - \lambda\|u_n\|^{q-2}\int_\Omega f|\overline{u}_n|^q = o_n(1). \quad (2.4.36)$$

Using (2.4.35) in (2.4.36), we get

$$\|\overline{u}_n\|^2 = 2^*_\alpha\|\overline{u}_n\|^2 + \frac{q-2.2^*_\alpha}{q}\lambda\|u_n\|^{q-2}\int_\Omega f(x)|\overline{u}|^q dx + o_n(1).$$

This implies that

$$\|\overline{u}_n\|^2 = \left(\frac{q-2.2^*_\alpha}{1-2^*_\alpha}\right)\frac{\lambda\|u_n\|^{q-2}}{q}\int_\Omega f(x)|\overline{u}|^q dx + o_n(1) \to 0 \text{ as } n \to \infty,$$

since $\|u_n\| \to \infty$ and $q < 2$. Which is a contradiction as $\|\overline{u}_n\| = 1$. Hence $\{u_n\}$ is bounded in $H_0^2(\Omega)$. □

Lemma 2.4.3. *Assume that* $c_\infty = \frac{N+4-\alpha}{2(2N-\alpha)}\|g^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} S_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}} - P_0\lambda^{\frac{2}{2-q}}$. *Then* $\mathcal{I}_\lambda$ *satisfies the* $(PS)_c$*-condition with* $c \in (-\infty, c_\infty)$*, where* P_0 *is same as given in Lemma 2.4.1.*

Proof. Let $\{u_n\} \subset H_0^2(\Omega)$ be a $(PS)_c$-sequence for $\mathcal{I}_\lambda$ with $c \in (-\infty, c_\infty)$. Then by Lemma 5.4.23, $\{u_n\}$ is bounded in $H_0^2(\Omega)$. Therefore up to subsequence, there exists $u \in H_0^2(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H_0^2(\Omega)$, $u_n \rightharpoonup u$ weakly in $L^{2^*}(\Omega)$, $u_n \to u$ strongly in $L^m(\Omega)$, for all $1 \leq m < 2^*$ and $u_n(x) \to u(x)$ pointwise a.e. in Ω. From $(f1)$, $(g1)$ and Lemma 2.4.1, we have $\mathcal{I}'_\lambda(u) = 0$ and

$$\lambda \int_\Omega f(x)|u_n|^q dx = \lambda \int_\Omega f(x)|u|^q dx + o_n(1).$$

Let $\tilde{u}_n = u_n - u$. Then $\tilde{u}_n \rightharpoonup 0$ weakly in $H_0^2(\Omega)$ and $\tilde{u}_n \to 0$ a.e. in Ω. By Brézis and Lieb Lemma [12] and Lemma 2.2.4, we have

$$\|\tilde{u}_n\|^2 = \|u_n\|^2 - \|u\|^2 + o_n(1). \tag{2.4.37}$$

$$\begin{aligned}\int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^{2^*_\alpha}|u_n(y)|^{2^*_\alpha}}{|x-y|^\alpha} &= \int_\Omega\int_\Omega g(x)g(y)\frac{|\tilde{u}_n(x)|^{2^*_\alpha}|\tilde{u}_n(x)|^{2^*_\alpha}}{|x-y|^\alpha} \\ &+ \int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha} + o_n(1). \end{aligned} \tag{2.4.38}$$

Now, using (2.4.37) and (2.4.38), we obtain

$$\frac{1}{2}\|\tilde{u}_n\|^2 - \frac{1}{2.2^*_\alpha}\int_\Omega\int_\Omega g(x)g(y)\frac{|\tilde{u}_n(x)|^{2^*_\alpha}|\tilde{u}_n(x)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy = c - \mathcal{I}_\lambda(u) + o_n(1) \tag{2.4.39}$$

and

$$\|\tilde{u}_n\|^2 - \int_\Omega\int_\Omega g(x)g(y)\frac{|\tilde{u}_n(x)|^{2^*_\alpha}|\tilde{u}_n(x)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy = \langle \mathcal{I}'_\lambda(u), u_n - u\rangle + o_n(1) = o_n(1).$$

Therefore, we may assume that

$$\|\tilde{u}_n\|^2 \to l, \text{ and } \int_\Omega\int_\Omega g(x)g(y)\frac{|\tilde{u}_n(x)|^{2^*_\alpha}|\tilde{u}_n(x)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy \to l. \tag{2.4.40}$$

Also by (2.2.3) and Lemma 2.2.1, we have

$$\|\tilde{u}_n\|^2 \geq S_{H,L}\left(\int_\Omega\int_\Omega \frac{|\tilde{u}_n(x)|^{2^*_\alpha}|\tilde{u}_n(x)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}}$$
$$\geq S_{H,L}\|g^+\|_\infty^{-\frac{2}{2^*_\alpha}}\left(\int_\Omega\int_\Omega g(x)g(y)\frac{|\tilde{u}_n(x)|^{2^*_\alpha}|\tilde{u}_n(x)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}}.$$

Combining this with (2.4.40), we obtain

$$l \geq S_{H,L}\|g^+\|_\infty^{-\frac{2}{2^*_\alpha}} l^{\frac{1}{2^*_\alpha}},$$

which implies

$$l\left(1 - S_{H,L}\|g^+\|_\infty^{-\frac{2}{2^*_\alpha}} l^{\frac{1}{2^*_\alpha}-1}\right) \geq 0.$$

Hence, either $l = 0$ or $l \geq \|g^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} S_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}}$. If $l = 0$, then proof is complete. Now, assume that if $l \geq \|g^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} S_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}}$, then equations (2.4.39), (2.4.40) and Lemma 2.4.1 infer that

$$c \geq \left(\frac{1}{2} - \frac{1}{2.2^*_\alpha}\right) l + \mathcal{I}_\lambda(u) \geq \frac{N+4-\alpha}{2(2N-\alpha)}\|g^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} S_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}} - P_0\lambda^{\frac{2}{2-q}} = c_\infty,$$

which gives a contradiction to the fact that $c \in (-\infty, c_\infty)$. Thus $l = 0$. Therefore, $u_n \to u$ strongly in $H_0^2(\Omega)$. This completes the proof of Lemma. □

2.5 Existence of a solution in $\mathcal{N}_\lambda^+$

Lemma 2.5.1. *If* $0 < \lambda < \Gamma_1$, *where* Γ_1 *is same as given in* (2.1.1), *then* $\mathcal{I}_\lambda$ *has a minimizer* u_λ *in* $\mathcal{N}_\lambda^+$ *which satisfies the following:*

(*i*) $\mathcal{I}_\lambda(u_\lambda) = \theta_\lambda = \theta_\lambda^+ < 0$;

(*ii*) u_λ *is a nontrivial solution of* (E_λ);

(*iii*) $\mathcal{I}_\lambda(u_\lambda) \to 0$ *as* $\lambda \to 0^+$.

Proof. By Lemma 2.3.9 (*i*), there exists a minimizing sequence $\{u_n\}$ for $\mathcal{I}_\lambda$ such that

$$\mathcal{I}_\lambda(u_n) = \theta_\lambda + o_n(1) \quad \text{and} \quad \mathcal{I}'_\lambda(u_n) = o_n(1).$$

The sequence $\{u_n\}$ is bounded in $H_0^2(\Omega)$, by coercivity of $\mathcal{I}_\lambda$. So up to subsequence, there exists $u_\lambda \in H_0^2(\Omega)$ such that $u_n \rightharpoonup u_\lambda$ weakly in $H_0^2(\Omega)$, $u_n \rightharpoonup u_\lambda$ weakly in $L^{2^*}(\Omega)$, $u_n \to u_\lambda$ strongly in $L^m(\Omega)$ for all $1 \leq m < 2^*$ and $u_n(x) \to u_\lambda(x)$ pointwise a.e. in Ω. Then, one can easily deduce that

$$|u_n|^{2^*_\alpha} \rightharpoonup |u_\lambda|^{2^*_\alpha} \text{ in } L^{\frac{2N}{2N-\alpha}}(\Omega) \text{ and } |u_n|^{2^*_\alpha-2}u_n \rightharpoonup |u_\lambda|^{2^*_\alpha-2}u_\lambda \text{ in } L^{\frac{2N}{N-\alpha+4}}(\Omega), \quad (2.5.41)$$

as $n \to \infty$. As we know that the Riesz potential defines a continuous linear map from $L^{\frac{2N}{2N-\alpha}}(\Omega)$ to $L^{\frac{2N}{\alpha}}(\Omega)$ so using Hardy-Littlewood-Sobolev inequality, we obtain

$$|x|^{-\alpha} * |u_n|^{2^*_\alpha} \rightharpoonup |x|^{-\alpha} * |u_\lambda|^{2^*_\alpha} \text{ weakly in } L^{\frac{2N}{\alpha}}(\Omega) \text{ as } n \to \infty. \quad (2.5.42)$$

Then, combining (2.5.41) and (2.5.42), we obtain that as $n \to \infty$,

$$\left(|x|^{-\alpha} * |u_n|^{2^*_\alpha}\right)|u_n|^{2^*_\alpha-2}u_n \rightharpoonup \left(|x|^{-\alpha} * |u_\lambda|^{2^*_\alpha}\right)|u_\lambda|^{2^*_\alpha-2}u_\lambda \text{ weakly in } L^{\frac{2N}{N+4}}(\Omega). \quad (2.5.43)$$

Since $\mathcal{I}'_\lambda(u_n) \to 0$ as $n \to \infty$, for any $\phi \in H_0^2(\Omega)$, we have

$$\lim_{n\to\infty}\left(\int_\Omega \Delta u_n \Delta\phi - \int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^{2^*_\alpha}|u_n(y)|^{2^*_\alpha-2}u_n(y)\phi(y)}{|x-y|^\alpha}dxdy - \lambda\int_\Omega f(x)|u_n|^{q-2}u_n\phi dx\right) = 0.$$

On passing the limit as $n \to \infty$, using (2.5.43) and continuity of g, we obtain

$$\int_\Omega \Delta u_\lambda \Delta\phi - \int_\Omega\int_\Omega g(x)g(y)\frac{|u_\lambda(x)|^{2^*_\alpha}|u_\lambda(y)|^{2^*_\alpha-2}u_\lambda(y)\phi(y)}{|x-y|^\alpha} = \lambda\int_\Omega f(x)|u_\lambda|^{q-2}u_\lambda\phi.$$

This implies that u_λ is a weak solution to the poblem (E_λ).

Since $u_n \in \mathcal{N}_\lambda$, we have

$$\lambda \int_\Omega f(x)|u_n|^q dx = \left(\frac{2.2^*_\alpha - 2}{2.2^*_\alpha - q} \right) \frac{q}{2} \|u_n\|^2 - \frac{2.2^*_\alpha q}{2.2^*_\alpha - q} \mathcal{I}_\lambda(u_n).$$

Taking $n \to \infty$, we have

$$\lambda \int_\Omega f(x)|u_\lambda|^q dx \geq -\frac{2.2^*_\alpha q}{2.2^*_\alpha - q} \theta_\lambda > 0, \tag{2.5.44}$$

since $\theta_\lambda < 0$. Hence u_λ is a nontrivial solution of (E_λ). Now our aim is to show that $u_n \to u_\lambda$ strongly in $H^2_0(\Omega)$ and $\mathcal{I}_\lambda(u_\lambda) = \theta_\lambda$. Using Fatou's Lemma, we deduce

$$\begin{aligned}
\theta_\lambda \leq \mathcal{I}_\lambda(u_\lambda) &= \frac{N+4-\alpha}{2(2N-\alpha)} \|u_\lambda\|^2 - \lambda \frac{2.2^*_\alpha - q}{2.2^*_\alpha q} \int_\Omega f(x)|u_\lambda|^q dx \\
&\leq \liminf_{n \to \infty} \left(\frac{N+4-\alpha}{2(2N-\alpha)} \|u_n\|^2 - \lambda \frac{2.2^*_\alpha - q}{2.2^*_\alpha q} \int_\Omega f(x)|u_n|^q dx \right) \\
&= \liminf_{n \to \infty} \mathcal{I}_\lambda(u_n) = \theta_\lambda.
\end{aligned}$$

Thus $\mathcal{I}_\lambda(u_\lambda) = \theta_\lambda$ gives $\lim\limits_{n \to \infty} \|u_n\|^2 = \|u_\lambda\|^2$. Now, Brézis and Lieb Lemma [12] implies $u_n \to u_\lambda$ strongly in $H^2_0(\Omega)$. Moreover, we have $u_\lambda \in \mathcal{N}^+_\lambda$. We show this by contradiction. Suppose $u_\lambda \in \mathcal{N}^-_\lambda$. Then, by Lemma 2.3.3$(ii)$ and (2.5.44), we get

$$\int_\Omega \int_\Omega g(x)g(y) \frac{|u_\lambda(x)|^{2^*_\alpha} |u_\lambda(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy > 0 \text{ and } \int_\Omega f(x)|u_\lambda|^q dx > 0.$$

Using Lemma 2.3.4 (iv), there are unique t^+ and t^- such that $t^+ u_\lambda \in \mathcal{N}^+_\lambda$ and $t^- u_\lambda \in \mathcal{N}^-_\lambda$. In particular, we have $t^+ < t^- = 1$. Since $\Phi'_{u_\lambda}(t^+) = 0$ and $\Phi''_{u_\lambda}(t^+) > 0$, there exists t^* satisfying $t^+ < t^* \leq t^-$ such that $\mathcal{I}_\lambda(t^+ u_\lambda) < \mathcal{I}_\lambda(t^* u_\lambda)$. Thus, from Lemma 2.3.4 (iv), we have

$$\mathcal{I}_\lambda(t^+ u_\lambda) < \mathcal{I}_\lambda(t^* u_\lambda) \leq \mathcal{I}_\lambda(t^- u_\lambda) = \mathcal{I}_\lambda(u_\lambda),$$

which is a contradiction. Hence $u_\lambda \in \mathcal{N}^+_\lambda$.

(iii) Now from Lemma 2.3.6(i) and (2.3.13), we obtain

$$0 > \theta_\lambda^+ \geq \theta_\lambda = \mathcal{I}_\lambda(u_\lambda) > -\frac{2.2_\alpha^* - q}{2.2_\alpha^* q}\lambda\|f^+\|_\infty|\Omega|^{\frac{2^*-q}{2^*}} S_{H,L}^{-\frac{q}{2}}(C(N,\alpha))^{-\frac{q}{2.2_\alpha^*}}\|u_\lambda\|^q,$$

which implies that $\mathcal{I}_\lambda(u_\lambda) \to 0$ as $\lambda \to 0^+$. □

Proof of Theorem 2.1.1: From Lemma 2.5.1, we conclude that (E_λ) has a non-trivial solution $u_\lambda \in \mathcal{N}_\lambda^+$ satisfying $\mathcal{I}_\lambda(u_\lambda) \to 0$ as $\lambda \to 0^+$. □

2.6 Existence of a solution in $\mathcal{N}_\lambda^-$

In this section, we show the existence of a second weak solution as a limit of Palais-Smale sequence which is obtained as minimizing sequence for $\mathcal{I}_\lambda$ in $\mathcal{N}_\lambda^-$.
Without loss of generality, we may assume that $0 \in \Omega$ and $B(0, 2\beta) \subset \Omega$. Let $\phi \in C_c^\infty(\Omega)$ be a fixed cut-off function such that $0 \leq \phi \leq 1$ in $\mathbb{R}^N$, $\phi(x) = 1$ on $B_\beta = B(0,\beta)$ and $\phi(x) = 0$ in $\mathbb{R}^N \setminus B_{2\beta}$ with $|\nabla\phi| \leq C, |\Delta\phi| \leq C$. Define

$$\overline{U}_\epsilon(x) = \phi U_\epsilon(x),$$

where $U_\epsilon(x)$ is define in (2.2.8). Then, we have the following norm estimates ([21]).

Lemma 2.6.1. *The following estimates hold.*

$$\|\overline{U}_\epsilon(x)\|^2 = S^{\frac{N}{4}} + o(\epsilon^{N-4}).$$

$$\int_\Omega |\overline{U}_\epsilon(x)|^{2^*} = S^{\frac{N}{4}} + o(\epsilon^N).$$

$$\int_\Omega |\overline{U}_\epsilon(x)|^q dx = \begin{cases} o\left(\epsilon^{\frac{N-4}{2}q}\right), & q < \frac{N}{N-4} \\ o\left(\epsilon^{N-\frac{N-4}{2}q}|\ln\epsilon|\right), & q = \frac{N}{N-4} \\ o\left(\epsilon^{N-\frac{N-4}{2}q}\right), & q > \frac{N}{N-4}. \end{cases} \tag{2.6.45}$$

Lemma 2.6.2. *Suppose that $(f1)-(g2)$ hold with $\frac{N}{N-4} \leq q < 2$ and there is $\bar{\Lambda} > 0$ such that $\lambda \in (0, \bar{\Lambda})$, then there exists $U_\lambda \in H_0^2(\Omega) \setminus \{0\}$ such that*

$$\sup_{t\geq 0}\mathcal{I}_\lambda(tU_\lambda) < c_\infty,$$

where c_∞ is the constant given in Lemma 2.4.3 and all C_i's with $1 \leq i \leq 12$ are positive constants. In particular, $\theta_\lambda^- < c_\infty$ for all $\lambda \in (0, \bar{\Lambda})$.

Proof. By assumption $(g2)$, there exists $0 < \beta \leq r_0$ such that for all $x \in B(0, 2\beta)$ with $\delta_0 > \frac{2N-\alpha}{2}$

$$g(x) = g(0) + o(|x|^{\delta_0}), \ \text{ as } \ x \to 0. \tag{2.6.46}$$

Define a functional $J : H_0^2(\Omega) \to \mathbb{R}$ such that

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2.2_\alpha^*}\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^\alpha}dxdy, \ \ \forall \ u \in H_0^2(\Omega).$$

$$R(u) = \frac{\|u\|^2}{\left(\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2_\alpha^*}}}. \tag{2.6.47}$$

Using Lemma 2.6.1 and (2.2.7), we obtain

$$\|\overline{U}_\epsilon(x)\|^2 = (C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}}S_{H,L}^{\frac{N}{4}} + o\left(\epsilon^{N-4}\right). \tag{2.6.48}$$

and

$$\int_\Omega |\overline{U}_\epsilon(x)|^{2^*} = (C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}}S_{H,L}^{\frac{N}{4}} + o\left(\epsilon^{N}\right). \tag{2.6.49}$$

Step 1: We will show that

$$\sup_{t\geq 0} J(t\overline{U}_\epsilon) \leq \frac{N+4-\alpha}{2(2N-\alpha)}\|g^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}}S_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}} + \begin{cases} o(\epsilon^{N-4}), & \alpha \leq 8 \\ o\left(\epsilon^{\frac{2N-\alpha}{2}}\right), & \alpha > 8. \end{cases}$$

Now, using Hardy-Littlewood-Sobolev inequality and (2.6.49), we have

$$\begin{aligned}\left(\int_\Omega\int_\Omega g(x)g(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}} &\le \|g^+\|_\infty^{\frac{2}{2^*_\alpha}}(C(N,\alpha))^{\frac{1}{2^*_\alpha}}\|\overline{U}_\epsilon(x)\|^2_{2^*}\\ &= \|g^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}}(C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}}S_{H,L}^{\frac{N-4}{4}}\\ &\quad + o(\epsilon^{N-4}).\end{aligned}$$

Thus

$$\int_\Omega\int_\Omega g(x)g(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy \le \|g^+\|^2_\infty(C(N,\alpha))^{\frac{N}{4}}S_{H,L}^{\frac{2N-\alpha}{4}} + o(\epsilon^{2N-\alpha}).$$

Consider

$$\begin{aligned}&\epsilon^{\alpha-2N}\|g^+\|^2_\infty(C(N,\alpha))^{\frac{N}{4}}S_{H,L}^{\frac{2N-\alpha}{4}} - \epsilon^{\alpha-2N}\int_\Omega\int_\Omega g(x)g(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\\ &= \mu^{2N-\alpha}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(g(0))^2}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}\\ &\qquad\qquad - \epsilon^{\alpha-2N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g(x)g(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha}\\ &= \mu^{2N-\alpha}\Bigg[\int_{\mathbb{R}^N\setminus B_\beta}\frac{g(0)\left(g(0)-g(x)|\phi(x)|^{2^*_\alpha}\right)}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}}\left(\int_{\mathbb{R}^N}\frac{1}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dy\right)dx\\ &\qquad + \int_{\mathbb{R}^N\setminus B_\beta}\frac{g(x)|\phi(x)|^{2^*_\alpha}}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}}\left(\int_{\mathbb{R}^N}\frac{g(0)-g(y)|\phi(y)|^{2^*_\alpha}}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dy\right)dx\\ &\qquad + \int_{B_\beta}\frac{g(0)\left(g(0)-g(x)\right)}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}}\left(\int_{\mathbb{R}^N}\frac{1}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dy\right)dx\\ &\qquad + \int_{B_\beta}\frac{g(x)}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}}\left(\int_{\mathbb{R}^N}\frac{g(0)-g(y)|\phi(y)|^{2^*_\alpha}}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dy\right)dx\Bigg]\\ &= D_1+D_2+D_3+D_4, \qquad (2.6.50)\end{aligned}$$

where $\mu = [N(N+2)(N-2)(N-4)]^{\frac{N-4}{8}}$.

Consider D_1, we have

$$\begin{aligned} D_1 = \mu^{2N-\alpha} \Bigg[& \int_{\mathbb{R}^N \setminus B_\beta} \frac{g(0)\left(g(0) - g(x)|\phi(x)|^{2^*_\alpha}\right)}{(\epsilon^2 + |x|^2)^{\frac{2N-\alpha}{2}}} \left(\int_{\mathbb{R}^N \setminus B_\beta} \frac{dy}{(\epsilon^2 + |y|^2)^{\frac{2N-\alpha}{2}} |x-y|^\alpha} \right) dx \\ & + \int_{\mathbb{R}^N \setminus B_\beta} \frac{g(0)\left(g(0) - g(x)|\phi(x)|^{2^*_\alpha}\right)}{(\epsilon^2 + |x|^2)^{\frac{2N-\alpha}{2}}} \left(\int_{B_\beta} \frac{dy}{(\epsilon^2 + |y|^2)^{\frac{2N-\alpha}{2}} |x-y|^\alpha} \right) dx \Bigg] \\ = D_{1,1} + D_{1,2}. & \end{aligned}$$

Applying Hardy-Littlewood-Sobolev inequality on $D_{1,1}$ and $D_{1,2}$ respectively yield

$$\begin{aligned} D_{1,1} &\le C_1 \left(\int_{\mathbb{R}^N \setminus B_\beta} \frac{dx}{(\epsilon^2 + |x|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \left(\int_{\mathbb{R}^N \setminus B_\beta} \frac{dy}{(\epsilon^2 + |y|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \\ &= C_1 \left(\int_{\mathbb{R}^N \setminus B_\beta} \frac{dt}{(\epsilon^2 + |t|^2)^N} \right)^{\frac{2N-\alpha}{N}} \\ &\le C_1 \left(\int_\beta^\infty \frac{r^{N-1}}{r^{2N}} dr \right)^{\frac{2N-\alpha}{N}} = C_2. \end{aligned}$$

and

$$\begin{aligned} D_{1,2} &\le C_3 \int_{\mathbb{R}^N \setminus B_\beta} \int_{B_\beta} \frac{1}{(\epsilon^2 + |x|^2)^{\frac{2N-\alpha}{2}} (\epsilon^2 + |y|^2)^{\frac{2N-\alpha}{2}} |x-y|^\alpha} dx dy \\ &\le C_3 \left(\int_{\mathbb{R}^N \setminus B_\beta} \frac{dx}{(\epsilon^2 + |x|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \left(\int_{B_\beta} \frac{dy}{(\epsilon^2 + |y|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \\ &\le C_4 \left(\int_{\mathbb{R}^N \setminus B_\beta} \frac{dx}{|x|^{2N}} \right)^{\frac{2N-\alpha}{2N}} \left(\int_0^\beta \frac{r^{N-1} dr}{(\epsilon^2 + r^2)^N} \right)^{\frac{2N-\alpha}{2N}} \\ &\le o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right) \left(\int_0^{\frac{\beta}{\epsilon}} \frac{t^{N-1} dt}{(1 + t^2)^N} \right)^{\frac{2N-\alpha}{2N}} \\ &\le o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right) \left(\int_0^\infty \frac{t^{N-1} dt}{(1 + t^2)^N} \right)^{\frac{2N-\alpha}{2N}} \\ &= o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right). \end{aligned}$$

Thus, $D_1 = C_2 + o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right)$.

Further on taking D_2, we obtain

$$\begin{aligned}
D_2 = \mu^{2N-\alpha} \Bigg[&\int_{\mathbb{R}^N \setminus B_\beta} \frac{g(x)|\phi(x)|^{2^*_\alpha}}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}} \left(\int_{\mathbb{R}^N \setminus B_\beta} \frac{g(0)-g(y)|\phi(y)|^{2^*_\alpha}}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha} dy \right) dx \\
&+ \int_{\mathbb{R}^N \setminus B_\beta} \frac{g(x)|\phi(x)|^{2^*_\alpha}}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}} \left(\int_{B_\beta} \frac{g(0)-g(y)|\phi(y)|^{2^*_\alpha}}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha} dy \right) dx \Bigg] \\
= D_{2,1} + D_{2,2}.
\end{aligned}$$

Now estimating $D_{2,1}$ same as $D_{1,1}$, we have

$$D_{2,1} \leq C_5 \left(\int_{\mathbb{R}^N \setminus B_\beta} \frac{dx}{(\epsilon^2+|x|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \left(\int_{\mathbb{R}^N \setminus B_\beta} \frac{dy}{(\epsilon^2+|y|^2)^N} \right)^{\frac{2N-\alpha}{2N}} = C_6.$$

Using Hardy-Littlewood-Sobolev inequality $D_{2,2}$ and (2.6.46), we get

$$\begin{aligned}
D_{2,2} &\leq C_7 \left(\int_{\mathbb{R}^N \setminus B_\beta} \frac{dx}{(\epsilon^2+|x|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \left(\int_{B_\beta} \frac{|y|^{\frac{2N\delta_0}{2N-\alpha}}}{(\epsilon^2+|y|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \\
&\leq C_8 \left(\int_{\mathbb{R}^N \setminus B_\beta} \frac{dx}{(\epsilon^2+|x|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \left(\int_{B_\beta} \frac{|y|^{\frac{2N\delta_0}{2N-\alpha}}}{|y|^{2N}} \right)^{\frac{2N-\alpha}{2N}} = C_9.
\end{aligned}$$

Hence, $D_2 = C_6 + C_9$.

For D_3, we use Hardy-Littlewood-Sobolev inequality with (2.6.46) which implies that

$$\begin{aligned}
D_3 = \mu^{2N-\alpha} \Bigg[&\int_{B_\beta} \int_{\mathbb{R}^N \setminus B_\beta} \frac{g(0)\left(g(0)-g(x)\right)}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha} dxdy \\
&+ \int_{B_\beta} \int_{B_\beta} \frac{g(0)\left(g(0)-g(x)\right)}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha} dxdy \Bigg] \\
= D_{3,1} + D_{3,2}.
\end{aligned}$$

Now,

$$
\begin{aligned}
D_{3,1} &\leq \mu^{2N-\alpha} \int_{B_\beta} \int_{\mathbb{R}^N \setminus B_\beta} \frac{g(0)|x|^{\delta_0}}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}} (\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}} |x-y|^\alpha} dxdy \\
&\leq \mu^{2N-\alpha} \left(\int_{B_\beta} \frac{|x|^{\frac{2N\delta_0}{2N-\alpha}} dx}{(\epsilon^2+|x|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \left(\int_{\mathbb{R}^N \setminus B_\beta} \frac{dy}{(\epsilon^2+|y|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \\
&\leq C_{10} \left(\int_{B_\beta} \frac{|x|^{\frac{2N\delta_0}{2N-\alpha}} dx}{|x|^{2N}} \right)^{\frac{2N-\alpha}{2N}} \left(\int_{\mathbb{R}^N \setminus B_\beta} \frac{dy}{(\epsilon^2+|y|^2)^N} \right)^{\frac{2N-\alpha}{2N}} = C_{11}.
\end{aligned}
$$

and

$$
\begin{aligned}
D_{3,2} &\leq \mu^{2N-\alpha} \int_{B_\beta} \int_{B_\beta} \frac{g(0)|x|^{\delta_0}}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}} (\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}} |x-y|^\alpha} dxdy \\
&\leq \mu^{2N-\alpha} \left(\int_{B_\beta} \frac{|x|^{\frac{2N\delta_0}{2N-\alpha}} dx}{(\epsilon^2+|x|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \left(\int_{B_\beta} \frac{dy}{(\epsilon^2+|y|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \\
&\leq o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right) \left(\int_{B_\beta} \frac{|x|^{\frac{2N\delta_0}{2N-\alpha}} dx}{|x|^{2N}} \right)^{\frac{2N-\alpha}{2N}} \left(\int_0^{\frac{\beta}{\epsilon}} \frac{r^{N-1} dr}{(1+r^2)^N} \right)^{\frac{2N-\alpha}{2N}} \\
&\leq o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right) \left(\int_0^\infty \frac{r^{N-1} dr}{r^{2N}} \right)^{\frac{2N-\alpha}{2N}} = o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right).
\end{aligned}
$$

Thus

$$
D_3 = C_{11} + o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right).
$$

Similarly on taking D_4, we have

$$
\begin{aligned}
D_4 = \mu^{2N-\alpha} \Bigg[&\int_{B_\beta} \frac{g(x)}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}} \left(\int_{\mathbb{R}^N \setminus B_\beta} \frac{g(0)-g(y)|\phi(y)|^{2^*_\alpha}}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}} |x-y|^\alpha} dy \right) dx \\
&+ \int_{B_\beta} \int_{B_\beta} \frac{g(x)\left(g(0)-g(y)\right)}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}} (\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}} |x-y|^\alpha} dxdy \Bigg] \\
= D_{4,1} + D_{4,2}.&
\end{aligned}
$$

By the same approach used in $D_{1,2}$ and $D_{3,2}$ respectively, we obtain

$$D_{4,1} = o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right) \text{ and } D_{4,2} = o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right).$$

Hence, $D_4 = o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right) + o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right) = o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right)$. Therefore

$$\begin{aligned} D_1 + D_2 + D_3 + D_4 &= C_2 + o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right) + C_6 + C_9 + o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right) + o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right) \\ &= \widehat{C} + o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right), \end{aligned}$$

where $\widehat{C} = C_2 + C_9 + C_{12}$. Further, using (2.6.50), we obtain

$$\begin{aligned} 0 &\leq \epsilon^{\alpha-2N}\|g^+\|_\infty^2 C(N,\alpha)^{\frac{N}{4}} S_{H,L}^{\frac{2N-\alpha}{4}} - \epsilon^{\alpha-2N}\int_\Omega\int_\Omega g(x)g(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha} \\ &\leq \widehat{C} + o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right). \end{aligned}$$

This implies that

$$\begin{aligned} 0 &\leq 1 - \|g^+\|_\infty^{-2}(C(N,\alpha))^{-\frac{N}{4}} S_{H,L}^{-\frac{2N-\alpha}{4}} \int_\Omega\int_\Omega g(x)g(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy \\ &\leq \epsilon^{2N-\alpha}\|g^+\|_\infty^{-2}(C(N,\alpha))^{-\frac{N}{4}} S_{H,L}^{-\frac{2N-\alpha}{4}}\widehat{C} + o\left(\epsilon^{\frac{2N-\alpha}{2}}\right). \end{aligned}$$

Furthermore

$$\begin{aligned} 0 &\leq 1 - \epsilon^{2N-\alpha}\|g^+\|_\infty^{-2}(C(N,\alpha))^{-\frac{N}{4}} S_{H,L}^{-\frac{2N-\alpha}{4}}\widehat{C} - o\left(\epsilon^{\frac{2N-\alpha}{2}}\right) \\ &\leq \|g^+\|_\infty^{-2}(C(N,\alpha))^{-\frac{N}{4}} S_{H,L}^{-\frac{2N-\alpha}{4}} \int_\Omega\int_\Omega g(x)g(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha} \leq 1. \end{aligned}$$

Now, choose $\epsilon > 0$ such that $\epsilon^{2N-\alpha}\|g^+\|_\infty^{-2}(C(N,\alpha))^{-\frac{N}{4}} S_{H,L}^{-\frac{2N-\alpha}{4}}\widehat{C} < 1$. Thus

$$\begin{aligned} 0 &\leq 1 - \epsilon^{2N-\alpha}\|g^+\|_\infty^{-2}(C(N,\alpha))^{-\frac{N}{4}} S_{H,L}^{-\frac{2N-\alpha}{4}}\widehat{C} - o\left(\epsilon^{\frac{2N-\alpha}{2}}\right) \\ &\leq \left(1 - \epsilon^{2N-\alpha}\|g^+\|_\infty^{-2}(C(N,\alpha))^{-\frac{N}{4}} S_{H,L}^{-\frac{2N-\alpha}{4}}\widehat{C} - o\left(\epsilon^{\frac{2N-\alpha}{2}}\right)\right)^{\frac{1}{2^*_\alpha}} \leq 1. \end{aligned}$$

Moreover

$$
\begin{aligned}
0 &\leq \|g^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}} (C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}} S_{H,L}^{\frac{N-4}{4}} - \epsilon^{2N-\alpha} \|g^+\|_\infty^{\frac{2(4-N+\alpha)}{2N-\alpha}} (C(N,\alpha))^{\frac{(\alpha-N-4)N}{(2N-\alpha)4}} S_{H,L}^{\frac{\alpha-N-4}{4}} \widehat{C} \\
&\qquad - o\left(\epsilon^{\frac{2N-\alpha}{2}}\right) \\
&\leq \left(\int_\Omega \int_\Omega g(x)g(y) \frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha} |\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy \right)^{\frac{1}{2^*_\alpha}} \leq \|g^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}} (C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}} S_{H,L}^{\frac{N-4}{4}}.
\end{aligned} \tag{2.6.51}
$$

With the help of (2.6.47), (2.6.48) and (2.6.51), we obtain

$$
\begin{aligned}
R(\overline{U}_\epsilon) &= \frac{\|\overline{U}_\epsilon\|^2}{\left(\int_\Omega \int_\Omega g(x)g(y) \frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha} |\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy \right)^{\frac{1}{2^*_\alpha}}} \\
&\leq \frac{(C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}} S_{H,L}^{\frac{N}{4}} + o(\epsilon^{N-4})}{\|g^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}} (C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}} S_{H,L}^{\frac{N-4}{4}} - o\left(\epsilon^{2N-\alpha}\right) - o\left(\epsilon^{\frac{2N-\alpha}{2}}\right)} \\
&\leq \frac{\|g^+\|_\infty^{-\frac{2(N-4)}{2N-\alpha}} S_{H,L} + o(\epsilon^{N-4})}{1 - o\left(\epsilon^{\frac{2N-\alpha}{2}}\right)}.
\end{aligned}
$$

Let $\gamma(t) = \frac{t^2}{2}A - \frac{t^{2.2^*_\alpha}}{2.2^*_\alpha}B$, for any $A, B > 0$. $\gamma'(t) = 0 = tA - t^{2.2^*_\alpha - 1}B \implies t = \left(\frac{A}{B}\right)^{\frac{1}{2.2^*_\alpha - 2}} = t^*$, $\gamma''(t^*) < 0$. So γ attains its maximum at t^*. Thus $\max_{t \geq 0} \gamma(t) = \dfrac{N+4-\alpha}{2(2N-\alpha)} \left(\dfrac{A}{B^{1/2^*_\alpha}}\right)^{\frac{2N-\alpha}{N+4-\alpha}}$. Therefore, we have

$$
\begin{aligned}
\sup_{t \geq 0} J(t\overline{U}_\epsilon) &= \frac{t^2}{2} \|\overline{U}_\epsilon\|^2 - \frac{t^{2.2^*_\alpha}}{2.2^*_\alpha} \int_\Omega \int_\Omega g(x)g(y) \frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha} |\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy \\
&= \frac{N+4-\alpha}{2(2N-\alpha)} (R(\overline{U}_\epsilon))^{\frac{2N-\alpha}{N+4-\alpha}} \\
&\leq \frac{N+4-\alpha}{2(2N-\alpha)} \left(\|g^+\|_\infty^{-\frac{2(N-4)}{2N-\alpha}} S_{H,L} \left(1 + o(\epsilon^{N-4}) + o(\epsilon^{\frac{2N-\alpha}{2}})\right) \right)^{\frac{2N-\alpha}{N+4-\alpha}} \\
&\leq \frac{N+4-\alpha}{2(2N-\alpha)} \|g^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} S_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}} + \begin{cases} o(\epsilon^{N-4}), & \alpha \leq 8 \\ o(\epsilon^{\frac{2N-\alpha}{2}}), & \alpha > 8. \end{cases}
\end{aligned}
$$

Step 2: Now, our aim is to prove that there exists $\bar{\Lambda} > 0$ such that

$$\sup_{t\geq 0} \mathcal{I}_\lambda(tU_\lambda) < \frac{N+4-\alpha}{2(2N-\alpha)} \|g^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} S_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}} - P_0 \lambda^{\frac{2}{2-q}}, \ \forall\, \lambda \in (0, \bar{\Lambda}).$$

Let $\delta_1 > 0$ be such that $0 < \lambda^{\frac{2}{2-q}} < \delta_1$ and

$$\frac{N+4-\alpha}{2(2N-\alpha)} \|g^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} S_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}} - P_0 \lambda^{\frac{2}{2-q}} > 0.$$

$$\begin{aligned}
\mathcal{I}_\lambda(t\overline{U}_\epsilon) &= \frac{t^2}{2}\|\overline{U}_\epsilon\|^2 - \frac{t^{2.2^*_\alpha}}{2.2^*_\alpha} \int_\Omega \int_\Omega g(x)g(y) \frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha} |\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha} - \lambda \frac{t^q}{q} \int_\Omega f(x) |\overline{U}_\epsilon(x)|^q \\
&= \frac{t^2}{2}\|\overline{U}_\epsilon\|^2 - \frac{t^{2.2^*_\alpha}}{2.2^*_\alpha} \int_{B_{2\beta}} \int_{B_{2\beta}} g(x)g(y) \frac{|U_\epsilon(x)|^{2^*_\alpha} |U_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha} - \lambda \frac{t^q}{q} \int_{B_{2\beta}} f(x) |U_\epsilon(x)|^q.
\end{aligned}$$

With the help of assumption $(f2)$ and $(g2)$, we can conclude that

$$\mathcal{I}_\lambda(t\overline{U}_\epsilon) \leq \frac{t^2}{2}\|\overline{U}_\epsilon\|^2, \ \ \forall\, t \geq 0,\ \lambda > 0.$$

So there exists $t_0 \in (0,1)$ such that

$$\sup_{0\leq t\leq t_0} \mathcal{I}_\lambda(t\overline{U}_\epsilon) < \frac{N+4-\alpha}{2(2N-\alpha)} \|g^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} S_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}} - P_0 \lambda^{\frac{2}{2-q}} = c_\infty, \ \ \forall\, 0 < \lambda^{\frac{2}{2-q}} < \delta_1.$$

Let $0 < \epsilon \leq \delta_2$, then

$$\begin{aligned}
\sup_{t\geq t_0} \mathcal{I}_\lambda(t\overline{U}_\epsilon) &= \sup_{t\geq t_0} \left(J(t\overline{U}_\epsilon) - \lambda \frac{t^q}{q} \int_\Omega f(x)|\overline{U}_\epsilon|^q dx \right) \\
&\leq \frac{N+4-\alpha}{2(2N-\alpha)} \|g^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} S_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}} + o(\epsilon^\nu) - \lambda \frac{a_0 t_0^q}{q} \int_\Omega |\overline{U}_\epsilon(x)|^q dx,
\end{aligned} \tag{2.6.52}$$

where $\nu = \min\{N-4, \frac{2N-\alpha}{2}\}$. On combining (2.6.52), (2.6.45) and $\frac{N}{N-4} \leq q < 2$, we have

$$\sup_{t\geq t_0}\mathcal{I}_\lambda(t\overline{U}_\epsilon)\leq \frac{N+4-\alpha}{2(2N-\alpha)}\|g^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}}S_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}}+o(\epsilon^\nu) - \frac{\lambda a_0}{q}\begin{cases} o(\epsilon^{N-\frac{N-4}{2}q}|\ln\epsilon|), \ q=\frac{N}{N-4} \\ o(\epsilon^{N-\frac{N-4}{2}q}), \ q>\frac{N}{N-4}. \end{cases} \tag{2.6.53}$$

Take $\epsilon=\left(\lambda^{\frac{2}{2-q}}\right)^{\frac{1}{\nu}}$. Using (2.6.53), we get

$$\sup_{t\geq t_0}\mathcal{I}_\lambda(t\overline{U}_\epsilon)\leq \frac{N+4-\alpha}{2(2N-\alpha)}\|g^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}}S_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}}+o\left(\lambda^{\frac{2}{2-q}}\right) - \frac{\lambda a_0}{q}\begin{cases} o\left(\left(\lambda^{\frac{2}{2-q}}\right)^{\frac{N}{2\nu}}|\ln\lambda^{\frac{2}{2-q}}|\right), \ q=\frac{N}{N-4} \\ o\left(\left(\lambda^{\frac{2}{2-q}}\right)^{\left(\frac{N}{\nu}-\frac{N-4}{2\nu}q\right)}\right), \ q>\frac{N}{N-4}. \end{cases}$$

Case(i): When $\alpha\leq 8$, then $\nu=N-4$.
If $q=\frac{N}{N-4}$, then we can find $\delta_3>0$ with $0<\lambda^{\frac{2}{2-q}}<\delta_3$ such that

$$o\left(\lambda^{\frac{2}{2-q}}\right)-\lambda\frac{a_0}{q}o\left(\lambda^{\frac{N}{(2-q)(N-4)}}|\ln\lambda^{\frac{2}{2-q}}|\right)<-P_0\lambda^{\frac{2}{2-q}},$$

as $\lambda\to 0$, $|\ln\lambda^{\frac{2}{2-q}}|\to+\infty$ and $\lambda.o\left(\lambda^{\frac{N}{(2-q)(N-4)}}\right)=o\left(\lambda^{\frac{2}{2-q}}\right)$.
If $q>\frac{N}{N-4}$, then we can find $\delta_3^*>0$ with $0<\lambda^{\frac{2}{2-q}}<\delta_3^*$ such that

$$o\left(\lambda^{\frac{2}{2-q}}\right)-\lambda\frac{a_0}{q}o\left(\left(\lambda^{\frac{2}{2-q}}\right)^{\left(\frac{N}{N-4}-\frac{q}{2}\right)}\right)<-P_0\lambda^{\frac{2}{2-q}},$$

as $1+\frac{2}{2-q}\left(\frac{N}{N-4}-\frac{q}{2}\right)<\frac{2}{2-q}$ for $q>\frac{N}{N-4}$. Now choose $\delta_4=\min\{\delta_1^{\frac{2-q}{2}},\delta_2^{\frac{(2-q)(N-4)}{2}},\delta_3^{\frac{2-q}{2}},(\delta_3^*)^{\frac{2-q}{2}}\}>0$ such that

$$\sup_{t\geq 0}\mathcal{I}_\lambda(t\overline{U}_\epsilon)<\frac{N+4-\alpha}{2(2N-\alpha)}\|g^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}}S_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}}-P_0\lambda^{\frac{2}{2-q}}, \ \text{ for } 0<\lambda<\delta_4.$$

Case(ii): When $\alpha>8$, then $\nu=\frac{2N-\alpha}{2}$.

If $q = \frac{N}{N-4}$, then we can find $\overline{\delta}_3 > 0$ with $0 < \lambda^{\frac{2}{2-q}} < \overline{\delta}_3$ such that

$$o\left(\lambda^{\frac{2}{2-q}}\right) - \lambda\frac{a_0}{q} o\left(\lambda^{\frac{2N}{(2-q)(2N-\alpha)}} |\ln \lambda^{\frac{2}{2-q}}|\right) < -P_0\lambda^{\frac{2}{2-q}},$$

as $\lambda \to 0$, $|\ln \lambda^{\frac{2}{2-q}}| \to +\infty$ and $\lambda . o\left(\lambda^{\frac{2N}{(2-q)(2N-\alpha)}}\right) = o\left(\lambda^{\frac{2}{2-q}}\right)$.

If $q > \frac{N}{N-4}$, then we can find $\overline{\delta}_3^* > 0$ with $0 < \lambda^{\frac{2}{2-q}} < \overline{\delta}_3^*$ such that

$$o\left(\lambda^{\frac{2}{2-q}}\right) - \lambda\frac{a_0}{q} o\left(\left(\lambda^{\frac{2}{2-q}}\right)^{\left(\frac{2N}{2N-\alpha} - \frac{N-4}{2N-\alpha}q\right)}\right) < -P_0\lambda^{\frac{2}{2-q}},$$

as $1 + \frac{2}{2-q}\left(\frac{2N}{2N-\alpha} - \frac{N-4}{2N-\alpha}q\right) < \frac{2}{2-q}$ for $q > \frac{N}{N-4}$.

Choosing $\delta_4^* = \min\{\delta_1^{\frac{2-q}{2}}, \delta_2^{\frac{(2-q)(2N-\alpha)}{2}}, (\overline{\delta}_3)^{\frac{2-q}{2}}, (\overline{\delta}_3^*)^{\frac{2-q}{2}}\} > 0$, we obtain

$$\sup_{t\geq 0} \mathcal{I}_\lambda(t\overline{U}_\epsilon) < \frac{N+4-\alpha}{2(2N-\alpha)} \|g^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} S_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}} - P_0\lambda^{\frac{2}{2-q}}, \;\; \text{for } 0 < \lambda < \delta_4^*.$$

Hence, set $\bar{\Lambda} = \min\{\delta_4, \delta_4^*\}$. Thus, we have

$$\sup_{t\geq 0} \mathcal{I}_\lambda(t\overline{U}_\epsilon) < \frac{N+4-\alpha}{2(2N-\alpha)} \|g^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} S_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}} - P_0\lambda^{\frac{2}{2-q}} = c_\infty, \;\; \text{for } 0 < \lambda < \bar{\Lambda}.$$

Step 3: We want to show that for $0 < \lambda < \bar{\Lambda}$, $\theta_\lambda^- < c_\infty$.

Now by assumption $(f2), (g2)$ and definition of $\overline{U}_\epsilon$,

$$\int_\Omega \int_\Omega g(x)g(y)\frac{|\overline{U}_\epsilon(x)|^{2_\alpha^*}|\overline{U}_\epsilon(y)|^{2_\alpha^*}}{|x-y|^\alpha} dxdy > 0 \;\text{ and }\; \int_\Omega f(x)|\overline{U}_\epsilon|^q dx > 0.$$

Then by Lemma 2.3.4 (ii), definition of θ_λ^- and result of Step 2, there exists $t_\epsilon\overline{U}_\epsilon \in \mathcal{N}_\lambda^-$. Moreover

$$\theta_\lambda^- \;\leq\; \mathcal{I}_\lambda(t_\epsilon\overline{U}_\epsilon) \;\leq\; \mathcal{I}_\lambda(t\overline{U}_\epsilon) < \frac{N+4-\alpha}{2(2N-\alpha)} \|g^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} S_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}} - P_0\lambda^{\frac{2}{2-q}} = c_\infty,$$

for each $0 < \lambda < \bar{\Lambda}$. On taking $\overline{U}_\epsilon = U_\lambda$, we obtained the desirable result. $\square$

Lemma 2.6.3. *Assume that* $(f1), (g1), (f2)$ *and* $(g2)$ *hold. Then there exists* $\Gamma_2 > 0$ *for* $0 < \lambda < \Gamma_2$ *such that* $\mathcal{I}_\lambda$ *has a minimizer* $v_\lambda \in \mathcal{N}_\lambda^-$ *with*
(i) $\mathcal{I}_\lambda(v_\lambda) = \theta_\lambda^-$*;*
(ii) v_λ *is a nontrivial solution of* (E_λ).
where $\Gamma_2 = \min\{\bar{\Lambda}, \frac{q}{2}\Gamma_1\}$ *and* $\bar{\Lambda}$, Γ_1 *are given same as in Lemma 2.6.2 and* (2.1.1).

Proof. By virtue of Lemma 2.3.9 *(ii)*, for $0 < \lambda < \frac{q}{2}\Gamma_1$, there exists a $(PS)_{\theta_\lambda^-}$-sequence $\{u_n\} \subset \mathcal{N}_\lambda^-$ in $H_0^2(\Omega)$ for $\mathcal{I}_\lambda$. By Lemma 2.4.3, Lemma 2.6.2 and Lemma 2.3.6 *(ii)*, $0 < \lambda < \bar{\Lambda}$, $\mathcal{I}_\lambda$ satisfies the $(PS)_{\theta_\lambda^-}$- condition and $\theta_\lambda^- > 0$. By Lemma 2.3.1, we obtain that $\{u_n\}$ is bounded in $H_0^2(\Omega)$. Hence, there exists a subsequence denoted same as $\{u_n\}$ and $v_\lambda \in \mathcal{N}_\lambda^-$ such that $u_n \to v_\lambda$ in $H_0^2(\Omega)$ and for $0 < \lambda < \Gamma_2$, $\mathcal{I}_\lambda(v_\lambda) = \theta_\lambda^- > 0$. Thus, in the last, by the same arguments used in Lemma 2.5.1, for $0 < \lambda < \Gamma_2$, we find that v_λ is a nontrivial solution of (E_λ). □

Proof of Theorem 2.1.2: By Lemma 2.5.1 and Lemma 2.6.2, we find that (E_λ) has two nontrivial solution $u_\lambda \in \mathcal{N}_\lambda^+$ and $v_\lambda \in \mathcal{N}_\lambda^-$ respectively. Since $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \phi$, this implies that u_λ and v_λ are distinct. Hence proof is complete. □

2.7 Conclusion

The main purpose of this chapter is to examine the biharmonic equation with the combination of sublinear and critical Choquard term involving sign-changing weight functions. A substantial amount of work has been done to establish the multiplicity results for semilinear, quasilinear elliptic equations involving sign-changing nonlinearity with Choquard nonlinear term. But there is no work related to biharomic equations involving Choquard type nonlinearity with sign-changing weight functions. This is the main motivation to study the biharmonic equations with critical Chaquard type nonlinearity.
Critical problems are difficult to study due to lack of compactness in the embedding from $H_0^2(\Omega)$ to $L^{2^*}(\Omega)$. This creates a lot of difficulties to pass the limit in this minimizing sequence. To overcome these difficulties, one needs to study the critical

level. This level is calculated with the help of minimizers. So, we prove the minimizers for $S_{H,L}$ in case of biharmonic operator, which provides the main contribution in the literature. With the help of these minimizers, we establish the existence and multiplicity of nontrivial solutions for the problem (E_λ) with respect to parameter λ.

3

Biharmonic System with Hartree-Type Critical Nonlinearity

In this chapter, we study the biharmonic system of equations involving critical Choquard term with sign-changing weight functions. Precisely, we extend the work of chapter 2 to the system of critical Choquard equation having concave-convex nonlinearity with sign-changing weight functions.

Thus we consider the following biharmonic Choquard system involving concave-convex nonlinearities with critical exponent and sign-changing weight functions

$$(\mathcal{D}_{\lambda,\mu}) \begin{cases} \Delta^2 u = \lambda F(x)|u|^{r-2}u + H(x)\left(\displaystyle\int_\Omega \frac{H(y)|v(y)|^{2^*_\alpha}}{|x-y|^\alpha}dy\right)|u|^{2^*_\alpha-2}u & \text{in } \Omega,\\ \Delta^2 v = \mu G(x)|v|^{r-2}v + H(x)\left(\displaystyle\int_\Omega \frac{H(y)|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dy\right)|v|^{2^*_\alpha-2}v & \text{in } \Omega,\\ u = v = \nabla u = \nabla v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 5$, $0 < \alpha < N$, $1 < r < 2$, $2^*_\alpha = \frac{2N-\alpha}{N-4}$ is the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality, Δ^2 denotes the biharmonic operator and λ, μ are the parameter such that $(\lambda, \mu) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$.

The weight functions F, G and H are satisfying the following assumptions:

($Z1$) F, $G \in L^\beta(\Omega)$ with $\beta = \frac{2^*}{2^*-r}$ and $2^* = \frac{2N}{N-4}$, $F^\pm = \max\{\pm F, 0\} \not\equiv 0$ in $\overline{\Omega}$ and $G^\pm = \max\{\pm G, 0\} \not\equiv 0$ in $\overline{\Omega}$.

($Z2$) $H \in L^\infty(\Omega)$ and $H^+ = \max\{H, 0\} \not\equiv 0$ in Ω.

In order to present our main results, we define the constant Υ_1 as

$$\Upsilon_1 := \left(\frac{22^*_\alpha - 2}{22^*_\alpha - r}\right)^{\frac{2}{2-r}} \left[\frac{2-r}{2(22^*_\alpha - r)}\|H^+\|_\infty^{-2}(\bar{S}_{H,L})^{2^*_\alpha}\right]^{\frac{1}{2^*_\alpha - 1}} S^{\frac{r}{2-r}},$$

where $\bar{S}_{H,L}$ is defined later.

Now we state our following main results.

Theorem 3.0.1. *If $1 \leq r < 2$, $0 < \alpha < N$ and $\lambda, \mu > 0$ satisfy $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$, then the system $(\mathcal{D}_{\lambda,\mu})$ has at least one nontrivial solution in $H^2_0(\Omega) \times H^2_0(\Omega)$.*

For multiplicity result, we need the following additive assumptions on F, G and H respectively:

($Z3$) There exist a_0, b_0 and $r_0 > 0$ such that $B(0, 2r_0) \subset \Omega$ and $F(x) \geq a_0$, $G(x) \geq b_0$ for all $x \in B(0, 2r_0)$.

$(Z4)$ There exists $\delta_0 > \frac{2N-\alpha}{2}$ such that $\|H^+\|_\infty = H(0) = \max_{x\in\overline{\Omega}} h(x)$, $H(x) > 0$ for all $x \in B(0, 2r_0)$ and

$$H(x) = H(0) + o\left(|x|^{\delta_0}\right) \text{ as } x \to 0.$$

Theorem 3.0.2. *If $1 \leq r < 2$, $0 < \alpha < N$ and $\lambda, \mu > 0$ satisfy $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_2$ (where $\Upsilon_2 \leq \Upsilon_1$) , then the system $(\mathcal{D}_{\lambda,\mu})$ has at least two nontrivial solution in $H_0^2(\Omega) \times H_0^2(\Omega)$. Moreover, the solutions corresponding to the system $(\mathcal{D}_{\lambda,\mu})$ are not semi-trivial.*

We use $\mathcal{H} := H_0^2(\Omega) \times H_0^2(\Omega)$ as a function space with standard norm $\|(u,v)\| = \left(\|u\|^2 + \|v\|^2\right)^{\frac{1}{2}}$ and $\|u\| = \left(\int_\Omega |\Delta u|^2 dx\right)^{\frac{1}{2}}$.

Definition 3.0.1. *A pair of functions $(u, v) \in \mathcal{H}$ is said to be a weak solution of the system $(\mathcal{D}_{\lambda,\mu})$ if for all $(\phi_1, \phi_2) \in \mathcal{H}$, the following holds*

$$\int_\Omega \Delta u \Delta \phi_1 dx + \int_\Omega \Delta v \Delta \phi_2 dx - \lambda \int_\Omega F(x)|u|^{r-2} u\phi_1 dx - \mu \int_\Omega G(x)|v|^{r-2} v\phi_2 dx$$
$$- \int_\Omega \int_\Omega H(x)H(y) \left(\frac{|v(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha - 2}u(y)\phi_1(y) + |u(x)|^{2^*_\alpha}|v(y)|^{2^*_\alpha-2}v(y)\phi_2(y)}{|x-y|^\alpha} \right) = 0.$$

3.1 Preliminary results and important estimates

Consider the best constant $\bar{S}_{H,L}$ given as

$$\bar{S}_{H,L} := \inf_{u\in\mathcal{H}\setminus\{(0,0)\}} \frac{\|(u,v)\|^2}{\left(\int_\Omega \int_\Omega \frac{|u(x)|^{2^*_\alpha}|v(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{2^*_\alpha}}}.$$

Now, we state an important lemma which is used to show the relation between $\bar{S}_{H,L}$ and $S_{H,L}$.

Lemma 3.1.1. *For $u, v \in L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N)$, $0 < \alpha < N$ and $s \in [2_\alpha, 2^*_\alpha]$, the following inequality holds true*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^s|v(y)|^s}{|x-y|^\alpha} \leq \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^s|u(y)|^s}{|x-y|^\alpha}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x)|^s|v(y)|^s}{|x-y|^\alpha}\right)^{\frac{1}{2}}.$$

Proof. The proof is similar as given in [37]. □

Afterwards, we build a relation, connecting $\bar{S}_{H,L}$ and $S_{H,L}$ by using the idea of [3].

Lemma 3.1.2. *Show that* $\bar{S}_{H,L} = 2S_{H,L}$.

Proof. Let $\{k_n\} \subset H_0^2(\Omega)$ be a minimizing sequence for $S_{H,L}$. Choose the sequences $\{u_n = sk_n\}$ and $\{v_n = tk_n\}$ in $H_0^2(\Omega)$, where $s, t > 0$. Then the definition of $\bar{S}_{H,L}$ implies that

$$\bar{S}_{H,L} \leq \frac{\|(u_n, v_n)\|^2}{\left(\int_\Omega\int_\Omega \frac{|u_n(x)|^{2^*_\alpha}|v_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}}} = \left(\frac{s}{t}+\frac{t}{s}\right)\frac{\|k_n\|^2}{\left(\int_\Omega\int_\Omega \frac{|k_n(x)|^{2^*_\alpha}|k_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}}}. \tag{3.1.1}$$

Further, define a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x) = x + \frac{1}{x}$. Then $f(\frac{s}{t}) = \frac{s}{t} + \frac{t}{s}$ and f achieves its minimum at $x_0 = 1$. Thus, we have

$$\min_{x\in\mathbb{R}^+} f(x) = f(x_0) = 2.$$

Now, choose s, t in such a way that $s = t$ and taking $n \to \infty$ in (3.1.1), we obtain

$$\bar{S}_{H,L} \leq 2S_{H,L}. \tag{3.1.2}$$

At the same time, let $\{(u_n, v_n)\}$ be a minimizing sequence of $\bar{S}_{H,L}$. Take $a_n = s_n v_n$ for some $s_n > 0$ such that $\int_\Omega\int_\Omega \frac{|u_n(x)|^{2^*_\alpha}|u_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy = \int_\Omega\int_\Omega \frac{|a_n(x)|^{2^*_\alpha}|a_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy.$ This together with Lemma 3.1.1 implies that

$$\begin{aligned}\int_\Omega\int_\Omega \frac{|u_n(x)|^{2^*_\alpha}|a_n(y)|^{2^*_\alpha}}{|x-y|^\alpha} &\leq \left(\int_\Omega\int_\Omega \frac{|u_n(x)|^{2^*_\alpha}|u_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}\right)^{\frac{1}{2}}\left(\int_\Omega\int_\Omega \frac{|a_n(x)|^{2^*_\alpha}|a_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}\right)^{\frac{1}{2}} \\ &= \int_\Omega\int_\Omega \frac{|u_n(x)|^{2^*_\alpha}|u_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}.\end{aligned}$$

Thus we have

$$
\begin{aligned}
&\frac{\|(u_n,v_n)\|^2}{\left(\int_\Omega\int_\Omega \frac{|u_n(x)|^{2^*_\alpha}|v_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}}} = s_n\frac{\|(u_n,v_n)\|^2}{\left(\int_\Omega\int_\Omega \frac{|u_n(x)|^{2^*_\alpha}|a_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}}}\\
&\geq s_n\frac{\|u_n\|^2}{\left(\int_\Omega\int_\Omega \frac{|u_n(x)|^{2^*_\alpha}|u_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}}} + s_n^{-1}\frac{\|a_n\|^2}{\left(\int_\Omega\int_\Omega \frac{|a_n(x)|^{2^*_\alpha}|a_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}}}\\
&\geq \left(s_n+s_n^{-1}\right)S_{H,L} \geq \gamma(x_0)S_{H,L}.
\end{aligned}
$$

Now passing the limit as $n\to\infty$, we obtain

$$
\bar{S}_{H,L} \geq 2S_{H,L}. \tag{3.1.3}
$$

We desire our result after combining (3.1.2) and (3.1.3). □

Now, we prove some estimates, which are useful to obtain the existence of a second solution. Without loss of generality, we may assume that $0\in\Omega$ and $B(0,2\gamma)\subset\Omega$. Let $\phi\in C_c^\infty(\Omega)$ be a fixed cut-off function such that $0\leq\phi\leq 1$ in $\mathbb{R}^N$, $\phi(x)=1$ on $B_\gamma = B(0,\gamma)$ and $\phi(x)=0$ in $\mathbb{R}^N\setminus B_{2\gamma}$ with $|\nabla\phi|\leq C, |\Delta\phi|\leq C$.

Define $\overline{U}_\epsilon(x) := \phi U_\epsilon(x)$, where $U_\epsilon(x) = \epsilon^{\frac{4-N}{2}}U\left(\frac{x}{\epsilon}\right)$ with

$$
U(x) = \frac{[N(N+2)(N-2)(N-4)]^{\frac{N-4}{8}}}{(1+|x|^2)^{\frac{N-4}{2}}}.
$$

Lemma 3.1.3. *For Choquard term, the following estimate is true:*

$$
\begin{aligned}
&0\leq \|H^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}}(C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}}S_{H,L}^{\frac{N-4}{4}} - o\left(\epsilon^{\frac{2N-\alpha}{2}}\right)\\
&\leq \left(\int_\Omega\int_\Omega H(x)H(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}} \leq \|H^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}}(C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}}S_{H,L}^{\frac{N-4}{4}}.
\end{aligned} \tag{3.1.4}
$$

Proof. The proof is directly follow by Lemma 2.6.2 of Chapter 2. □

Now, we define the energy functional $I_{\lambda,\mu}:\mathcal{H}\to\mathbb{R}$ associated with $(\mathcal{D}_{\lambda,\mu})$ as

$$
\begin{aligned}
I_{\lambda,\mu}(u,v) = \frac{1}{2}\|(u,v)\|^2 - \frac{1}{r}\int_\Omega(\lambda F(x)|u|^r + \mu G(x)|v|^r) \\
- \frac{1}{2^*_\alpha}\int_\Omega\int_\Omega H(x)H(y)\left(\frac{|u(x)|^{2^*_\alpha}|v(y)|^{2^*_\alpha}}{|x-y|^\alpha}\right)dxdy. \qquad (3.1.5)
\end{aligned}
$$

Then $I_{\lambda,\mu}(u,v)$ is C^1 function on $\mathcal{H}$. Moreover, critical points of the functional $I_{\lambda,\mu}$ are the solutions of $(\mathcal{D}_{\lambda,\mu})$. For convenience, we denote $P_{\lambda,\mu}(u,v)$ and $Q(u,v)$ as

$$
\begin{aligned}
P_{\lambda,\mu}(u,v) &:= \int_\Omega(\lambda F(x)|u|^r + \mu G(x)|v|^r)dx, \\
Q(u,v) &:= \int_\Omega\int_\Omega H(x)H(y)\left(\frac{|u(x)|^{2^*_\alpha}|v(y)|^{2^*_\alpha}}{|x-y|^\alpha}\right)dxdy,
\end{aligned}
$$

throughout this chapter. Then using the Hölder's inequality, Sobolev embedding theorem, and the definition of $\bar{S}_{H,L}$, one can easily obtain

$$
\begin{aligned}
P_{\lambda,\mu}(u,v) \leq& S^{-\frac{r}{2}}\left(\lambda\|F\|_\beta\|u\|^r + \mu\|G\|_\beta\|v\|^r\right) \\
\leq& S^{-\frac{r}{2}}\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}}\|(u,v)\|^r. \qquad (3.1.6)
\end{aligned}
$$

and

$$
Q(u,v) \leq \|H^+\|^2_\infty(\bar{S}_{H,L})^{-2^*_\alpha}\|(u,v)\|^{22^*_\alpha}. \qquad (3.1.7)
$$

3.2 Analysis of Palais-Smale condition

In this section, firstly we show that $\mathcal{I}_\lambda$ satisfies the Palais-Smale condition below a certain level i.e. c_∞, which is used to prove the existence of second solution.

Lemma 3.2.1. *Consider $(Z1)$ and $(Z2)$ are true. Suppose $\{(u_n, v_n\} \subset \mathcal{H}$ is a $(PS)_c$-sequence for $I_{\lambda,\mu}$ such that $(u_n, v_n) \rightharpoonup (u,v)$ weakly in $\mathcal{H}$. Then $I'_{\lambda,\mu}(u,v) = 0$. Furthermore, there exists a positive constant K_0 depending on $r, \alpha, N, 2^*_\alpha$ and S such that*

$$I_{\lambda,\mu}(u,v) \geq -K_0 \left((\lambda \|F\|_\beta)^{\frac{2}{2-r}} + (\mu \|G\|_\beta)^{\frac{2}{2-r}} \right),$$

where $K_0 = \left(\frac{22^*_\alpha - r}{22^*_\alpha}\right)\left(\frac{2-r}{2}\right)\left[\left(\frac{2N-\alpha}{N+4-\alpha}\right)\left(\frac{22^*_\alpha - r}{22^*_\alpha}\right) S^{-\frac{r}{2}}\right]^{\frac{r}{2-r}} S^{-\frac{r}{2}}$.

Proof. If $\{(u_n, v_n)\}$ be a $(PS)_c$-sequence for $I_{\lambda,\mu}$ with $(u_n, v_n) \rightharpoonup (u, v)$ weakly in $\mathcal{H}$, then by using the standard argument, we get $I'_{\lambda,\mu}(u, v) = 0$. i.e.

$$\|(u,v)\|^2 - P_{\lambda,\mu}(u,v) - 2Q(u,v) = 0.$$

Above with Hölder's inequality, Sobolev embedding theorem and Young's inequality in (3.1.5) implies that

$$\begin{aligned} I_{\lambda,\mu}(u,v) =& \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right)\|(u,v)\|^2 - \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right)\int_\Omega (\lambda F(x)|u|^r + \mu G(x)|v|^r)\,dx \\ \geq& \frac{N+4-\alpha}{2(2N-\alpha)}\|(u,v)\|^2 - \left(\frac{22^*_\alpha - r}{22^*_\alpha}\right) S^{-\frac{r}{2}} \\ & \times \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} \right)^{\frac{2-r}{2}} \|(u,v)\|^r \\ \geq& \frac{N+4-\alpha}{2(2N-\alpha)}\|(u,v)\|^2 - \left(\frac{22^*_\alpha - r}{22^*_\alpha}\right) S^{-\frac{r}{2}} \\ & \times \left[\frac{2-r}{2} l^{\frac{2}{2-r}} \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} \right) + \frac{r}{2} l^{-\frac{2}{r}} \|(u,v)\|^2 \right] \\ =& -K_0 \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} \right), \end{aligned}$$

where, $K_0 = \left(\frac{22^*_\alpha - r}{22^*_\alpha}\right)\left(\frac{2-r}{2}\right)\left[\left(\frac{2N-\alpha}{N+4-\alpha}\right)\left(\frac{22^*_\alpha - r}{22^*_\alpha}\right) S^{-\frac{r}{2}}\right]^{\frac{r}{2-r}} S^{-\frac{r}{2}}$ and
$l = \left[\left(\frac{2N-\alpha}{N+4-\alpha}\right)\left(\frac{22^*_\alpha - r}{22^*_\alpha}\right) S^{-\frac{r}{2}}\right]^{\frac{r}{2}}$. This completes the proof. □

Lemma 3.2.2. *Assume $\{(u_n, v_n)\} \subset \mathcal{H}$ is a $(PS)_c$-sequence for $I_{\lambda,\mu}$, then $\{(u_n, v_n)\}$ is bounded in $\mathcal{H}$.*

Proof. Let $\{(u_n, v_n)\}$ be a $(PS)_c$-sequence for $I_{\lambda,\mu}$ in $\mathcal{H}$, then as per the definition of $(PS)_c$-sequence, $I_{\lambda,\mu}(u_n, v_n) \to c$ and $I'_{\lambda,\mu}(u_n, v_n) \to 0$ in $\mathcal{H}^{-1}$ i.e.

$$\frac{1}{2}\|(u_n,v_n)\|^2 - \frac{1}{r}P_{\lambda,\mu}(u_n,v_n) - \frac{1}{2^*_\alpha}Q(u_n,v_n) = c + o_n(1), \tag{3.2.8}$$

$$\|(u_n,v_n)\|^2 - P_{\lambda,\mu}(u_n,v_n) - Q(u_n,v_n) = o_n(1). \tag{3.2.9}$$

Now, our aim is to show that $\{(u_n,v_n)\}$ is bounded. On contrary, assume that $\|(u_n,v_n)\| \to \infty$ as $n \to \infty$ and take $(\widehat{u}_n,\widehat{v}_n) := \frac{(u_n,v_n)}{\|(u_n,v_n)\|}$. It follows that $\{(\widehat{u}_n,\widehat{v}_n)\}$ is a bounded sequence. Consequently, up to a subsequence $(\widehat{u}_n,\widehat{v}_n) \rightharpoonup (\widehat{u},\widehat{v})$ weakly in $\mathcal{H}$, $(\widehat{u}_n,\widehat{v}_n) \to (\widehat{u},\widehat{v})$ strongly in $L^m(\Omega)$ for all $1 \le m < 2^*$ and $(\widehat{u}_n(x),\widehat{v}_n(x)) \to (\widehat{u}(x),\widehat{v}(x))$ pointwise a.e. in $\Omega \times \Omega$.
Using (3.2.8) and (3.2.9), we have

$$\frac{\|(\widehat{u}_n,\widehat{v}_n)\|^2}{2} - \frac{\|(u_n,v_n)\|^{r-2}}{r}P_{\lambda,\mu}(\widehat{u}_n,\widehat{v}_n) - \frac{\|(u_n,v_n)\|^{22^*_\alpha-2}}{2^*_\alpha}Q(\widehat{u}_n,\widehat{v}_n) = o_n(1), \tag{3.2.10}$$

and

$$\|(\widehat{u}_n,\widehat{v}_n)\|^2 - \|(u_n,v_n)\|^{r-2}P_{\lambda,\mu}(\widehat{u}_n,\widehat{v}_n) - \|(u_n,v_n)\|^{22^*_\alpha-2}Q(\widehat{u}_n,\widehat{v}_n) = o_n(1). \tag{3.2.11}$$

From (3.2.10) and (3.2.11), we can deduce that

$$\|(\widehat{u}_n,\widehat{v}_n)\|^2 = \frac{2(2^*_\alpha - r)}{r(2^*_\alpha - 2)}\|(u_n,v_n)\|^{r-2}P_{\lambda,\mu}(\widehat{u}_n,\widehat{v}_n) + o_n(1). \tag{3.2.12}$$

Since $1 \le r < 2$ and $\|(u_n,v_n)\| \to \infty$, then (3.2.12) implies $\|(\widehat{u}_n,\widehat{v}_n)\|^2 \to 0$ as $n \to \infty$, which is a contradiction to the fact that $\|(\widehat{u}_n,\widehat{v}_n)\| = 1$. Thus, proof is completed. □

Lemma 3.2.3. *There exists*

$$c_\infty := \frac{N+4-\alpha}{2(2N-\alpha)}\left(\frac{\|H^+\|_\infty^{-2}}{2}\right)^{\frac{N-4}{N+4-\alpha}} \bar{S}_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}} - K_0\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right),$$

such that the energy functional $I_{\lambda,\mu}$ satisfies the $(PS)_c$-condition with $c \in (-\infty, c_\infty)$ and K_0 is defined in Lemma 3.2.1.

Proof. Let $\{(u_n, v_n)\} \subset \mathcal{H}$ be a $(PS)_c$-sequence for $I_{\lambda,\mu}$ with $0 < c < c_\infty$. Then by Lemma 3.2.2, $\{(u_n, v_n)\}$ is a bounded sequence in $\mathcal{H}$. Thus, up to a subsequence, $(u_n, v_n) \rightharpoonup (u, v)$ weakly in $\mathcal{H}$. So $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ weakly in $H_0^2(\Omega)$, $u_n \to u$ and $v_n \to v$ strongly in $L^m(\Omega)$ for all $1 \leq m < 2^*$ and $u_n \to u$, $v_n \to v$ pointwise a.e. in Ω. Therefore

$$P_{\lambda,\mu}(u_n, v_n) = P_{\lambda,\mu}(u, v) + o_n(1). \tag{3.2.13}$$

Also, $I'_{\lambda,\mu}(u, v) = 0$, follows from Lemma 3.2.1. Now, define $(\tilde{u}_n, \tilde{v}_n)$, where $\tilde{u}_n = u_n - u$, $\tilde{v}_n = v_n - v$. Then by Brézis-Lieb Lemma [12] and Lemma 2.2.4, we have

$$\begin{aligned} \|(\tilde{u}_n, \tilde{v}_n)\|^2 &= \|(u_n, v_n)\|^2 - \|(u, v)\|^2 + o_n(1), \\ Q(u_n, v_n) &= Q(\tilde{u}_n, \tilde{v}_n) + Q(u, v) + o_n(1). \end{aligned} \tag{3.2.14}$$

Using $I_{\lambda,\mu}(u_n, v_n) = c + o_n(1)$, $I'_{\lambda,\mu}(u_n, v_n) = o_n(1)$ and (3.2.13)-(3.2.14), we obtain

$$\frac{1}{2}\|(\tilde{u}_n, \tilde{v}_n)\|^2 - \frac{1}{2^*_\alpha} Q(\tilde{u}_n, \tilde{v}_n) = c - I_{\lambda,\mu}(u, v) + o_n(1), \tag{3.2.15}$$

and

$$\|(\tilde{u}_n, \tilde{v}_n)\|^2 - 2Q(\tilde{u}_n, \tilde{v}_n) = \langle I'_{\lambda,\mu}(u, v), (u_n - u, v_n - v)\rangle + o_n(1) = o_n(1).$$

Therefore, we may assume that

$$\|(\tilde{u}_n, \tilde{v}_n)\|^2 \to d, \text{ and } 2\int_\Omega \int_\Omega H(x)H(y) \frac{|\tilde{u}_n(x)|^{2^*_\alpha} |\tilde{v}_n(y)|^{2^*_\alpha}}{|x - y|^\alpha} \to d. \tag{3.2.16}$$

It follows from the definition of $\bar{S}_{H,L}$ that

$$\begin{aligned} \|(\tilde{u}_n, \tilde{v}_n)\|^2 &\geq \bar{S}_{H,L} \left(\int_\Omega \int_\Omega \frac{|\tilde{u}_n(x)|^{2^*_\alpha} |\tilde{v}_n(y)|^{2^*_\alpha}}{|x - y|^\alpha} \right)^{\frac{1}{2^*_\alpha}} \\ &\geq \bar{S}_{H,L} \|H^+\|_\infty^{-\frac{2}{2^*_\alpha}} \left(\int_\Omega \int_\Omega H(x)H(y) \frac{|\tilde{u}_n(x)|^{2^*_\alpha} |\tilde{v}_n(y)|^{2^*_\alpha}}{|x - y|^\alpha} \right)^{\frac{1}{2^*_\alpha}}. \end{aligned} \tag{3.2.17}$$

On combining (3.2.16) and (3.2.17), we have

$$d \geq \bar{S}_{H,L} \|H^+\|_\infty^{-\frac{2}{2^*_\alpha}} \left(\frac{d}{2}\right)^{\frac{1}{2^*_\alpha}},$$

which gives either $d = 0$ or $d \geq \left(\frac{\|H^+\|_\infty^{-\frac{2}{2^*_\alpha}}}{2}\right)^{\frac{N-4}{N+4-\alpha}} \bar{S}_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}}$. Further, if $d = 0$ then proof is complete. If $d \geq \left(\frac{\|H^+\|_\infty^{-\frac{2}{2^*_\alpha}}}{2}\right)^{\frac{N-4}{N+4-\alpha}} \bar{S}_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}}$, then according to (3.2.15), (3.2.16) and Lemma 3.2.1, we get

$$\begin{aligned} c &= \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right) d + I_{\lambda,\mu}(u,v) \\ &\geq \frac{N+4-\alpha}{2(2N-\alpha)} \left(\frac{\|H^+\|_\infty^{-2}}{2}\right)^{\frac{N-4}{N+4-\alpha}} \bar{S}_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}} - K_0 \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right) := c_\infty, \end{aligned}$$

a contradiction to $c < c_\infty$. Hence, $d = 0$ and with this we end the proof. □

3.3 The fibering map analysis

Now, we elaborate some important results for Nehari manifold and analysis of fibering map on $I_{\lambda,\mu}$. Notice that the energy functional $I_{\lambda,\mu}$ is unbounded below on $\mathcal{H}$. So we restrict $I_{\lambda,\mu}$ on an appropriate subset $\mathcal{N}_{\lambda,\mu}$ of $\mathcal{H}$, called Nehari manifold and defined as

$$\mathcal{N}_{\lambda,\mu} := \left\{(u,v) \in \mathcal{H} \setminus \{(0,0)\} : \langle I'_{\lambda,\mu}(u,v), (u,v)\rangle = 0\right\}.$$

Thus, $(u,v) \in \mathcal{N}_{\lambda,\mu}$ if and only if

$$\langle I'_{\lambda,\mu}(u,v), (u,v)\rangle = \|(u,v)\|^2 - P_{\lambda,\mu}(u,v) - 2Q(u,v) = 0. \tag{3.3.18}$$

Next, we see that $I_{\lambda,\mu}$ is bounded from below on $\mathcal{N}_{\lambda,\mu}$ in the following lemma.

Lemma 3.3.1. *The energy functional $I_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$.*

Proof. Let $(u,v) \in \mathcal{N}_{\lambda,\mu}$, then using (3.3.18) and (3.1.6), we have

$$\begin{aligned} I_{\lambda,\mu}(u,v) &= \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right) \|(u,v)\|^2 - \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right) P_{\lambda,\mu}(u,v) \\ &\geq \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right) \|(u,v)\|^2 - \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right) S^{-\frac{r}{2}} \\ &\qquad \times \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} \right)^{\frac{2-r}{2}} \|(u,v)\|^r. \end{aligned} \tag{3.3.19}$$

Since $1 < r < 2$. Therefore, $I_{\lambda,\mu}$ is coercive.

Now, consider the function $\varrho : \mathbb{R} \to \mathbb{R}$ as $\varrho(t) = b_1 t^2 - b_2 t^r$. Then one can easily see that $\varrho'(t) = 0$ if and only if $t = \left(\frac{b_2 r}{2b_1}\right)^{\frac{1}{2-r}} := t^*$ and $\varrho''(t^*) > 0$. So ϱ attains its minimum at t^*. Moreover,

$$\varrho(t) \geq \varrho(t^*) := -(2-r)\left(\frac{b_2}{2}\right)^{\frac{2}{2-r}} \left(\frac{r}{b_1}\right)^{\frac{r}{2-r}}.$$

Taking $b_1 = \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right)$, $b_2 = \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right) S^{-\frac{r}{2}} \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} \right)^{\frac{2-r}{2}}$ and $t = \|(u,v)\|$ in the function ϱ, we obtain

$$I_{\lambda,\mu}(u,v) \geq \varrho(\|(u,v)\|) \geq \varrho(t^*).$$

which yields the required assertion. $\square$

The Nehari manifold is intently related to the behaviour of fibering map $\Psi_{u,v} : t \to I_{\lambda,\mu}(tu,tv)$ for $t > 0$, defined as

$$\Psi_{u,v}(t) := I_{\lambda,\mu}(tu,tv) = \frac{t^2}{2}\|(u,v)\|^2 - \frac{t^r}{r} P_{\lambda,\mu}(u,v) - \frac{t^{22^*_\alpha}}{2^*_\alpha} Q(u,v).$$

Thus, $(tu,tv) \in \mathcal{N}_{\lambda,\mu}$ iff $\Psi'_{u,v}(t) = 0$. Furthermore

$$\begin{aligned} \Psi'_{u,v}(t) &= t\|(u,v)\|^2 - t^{r-1} P_{\lambda,\mu}(u,v) - 2t^{2^*_\alpha - 1} Q(u,v), \\ \Psi''_{u,v}(t) &= \|(u,v)\|^2 - (r-1)t^{r-2} P_{\lambda,\mu}(u,v) - 2(22^*_\alpha - 1)t^{22^*_\alpha - 2} Q(u,v). \end{aligned}$$

In particular, $(u,v) \in \mathcal{N}_{\lambda,\mu}$ if and only if $\Psi'_{u,v}(1) = 0$. Therefore it is obvious to split $\mathcal{N}_{\lambda,\mu}$ into three parts namely $\mathcal{N}^+_{\lambda,\mu}$, $\mathcal{N}^-_{\lambda,\mu}$ and $\mathcal{N}^0_{\lambda,\mu}$ corresponding to local minima, local maxima and point of inflexion respectively

$$\mathcal{N}^{\pm}_{\lambda,\mu} := \left\{(u,v) \in \mathcal{N}_{\lambda,\mu} : \Psi''_{u,v}(1) \gtrless 0\right\},\ \mathcal{N}^0_{\lambda,\mu} := \left\{(u,v) \in \mathcal{N}_{\lambda,\mu} : \Psi''_{u,v}(1) = 0\right\}.$$

We note that, for $(u,v) \in \mathcal{N}_{\lambda,\mu}$, we have

$$\Psi''_{u,v}(1) = \begin{cases} (2 - 22^*_\alpha)\|(u,v)\|^2 - (r - 22^*_\alpha)P_{\lambda,\mu}(u,v) \\ (2 - r)\|(u,v)\|^2 - 2(22^*_\alpha - r)Q(u,v). \end{cases} \tag{3.3.20}$$

Lemma 3.3.2. *If (u,v) is the local minimizer for $I_{\lambda,\mu}$ on subset of $\mathcal{N}_{\lambda,\mu}$, namely $\mathcal{N}^+_{\lambda,\mu}$ or $\mathcal{N}^-_{\lambda,\mu}$ such that $(u,v) \notin \mathcal{N}^0_{\lambda,\mu}$. Then $I'_{\lambda,\mu}(u,v) = 0$ in $\mathcal{H}^{-1}$, where $\mathcal{H}^{-1}$ denotes the dual space of $\mathcal{H}$.*

Proof. The proof follows the same as done in Lemma 1.3.1 in chapter 1. □

Lemma 3.3.3. *The following hold:*
(i) If $(u,v) \in \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu}$, then $P_{\lambda,\mu}(u,v) > 0$.
(ii) If $(u,v) \in \mathcal{N}^-_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu}$, then $Q(u,v) > 0$.

Proof. The proof is directly followed by (3.3.20). □

Before analyzing the fibering map, we define a map $\mathcal{S}_{u,v} : \mathbb{R}^+ \to \mathbb{R}$ such that

$$\mathcal{S}_{u,v}(t) := t^{2-r}\|(u,v)\|^2 - 2t^{22^*_\alpha - r}Q(u,v). \tag{3.3.21}$$

It is noted that for $t > 0$, $(tu, tv) \in \mathcal{N}_{\lambda,\mu}$ if and only if $\mathcal{S}_{u,v}(t) = P_{\lambda,\mu}(u,v)$. We will check the behaviour of $\mathcal{S}_{u,v}$ near 0 and $+\infty$. Since $1 < r < 2$ and $2 < 22^*_\alpha$, this implies that $\lim\limits_{t \to 0^+} \mathcal{S}_{(u,v)}(t) = 0$ and $\lim\limits_{t \to +\infty} \mathcal{S}_{u,v}(t) = -\infty$. Moreover, for critical points

$$\mathcal{S}'_{u,v}(t) = (2 - r)t^{1-r}\|(u,v)\|^2 - 2(22^*_\alpha - r)t^{22^*_\alpha - r - 1}Q(u,v).$$

One can easily see that $\mathcal{S}'_{u,v}(t) = 0$ if and only if $t = t_{\max}$, where

$$t_{\max} = \left(\frac{(2-r)\|(u,v)\|^2}{2(22^*_\alpha - r)Q(u,v)} \right)^{\frac{1}{22^*_\alpha - 2}}.$$

Also, $\mathcal{S}''_{u,v}(t) = (2-r)(1-r)t^{-r}\|(u,v)\|^2 - 2(22^*_\alpha - r)(22^*_\alpha - r - 1)t^{22^*_\alpha - r - 2}Q(u,v)$.

$$\begin{aligned}
\mathcal{S}''_{u,v}(t_{\max}) &= (2-r)(1-r)\left(\frac{2(22^*_\alpha - r)Q(u,v)}{(2-r)\|(u,v)\|^2} \right)^{\frac{r}{22^*_\alpha - 2}} \|(u,v)\|^2 \\
&\quad - 2(22^*_\alpha - r)(22^*_\alpha - r - 1)\left(\frac{(2-r)\|(u,v)\|^2}{2(22^*_\alpha - r)Q(u,v)} \right)^{\frac{22^*_\alpha - r - 2}{22^*_\alpha - 2}} Q(u,v) \\
&= \frac{\|(u,v)\|^{\frac{2(22^*_\alpha - r - 2)}{22^*_\alpha - 2}}}{(Q(u,v))^{-\frac{r}{22^*_\alpha - 2}}} \Bigg[(2-r)(1-r)\left(\frac{2(22^*_\alpha - r)}{2-r} \right)^{\frac{r}{22^*_\alpha - 2}} \\
&\quad -2(22^*_\alpha - r)(22^*_\alpha - r - 1)\left(\frac{2-r}{2(22^*_\alpha - r)} \right)\left(\frac{2(22^*_\alpha - r)}{2-r} \right)^{\frac{r}{22^*_\alpha - 2}} \Bigg] \\
&= \frac{(2 - 22^*_\alpha)\left(2(22^*_\alpha - r)\right)^{\frac{r}{22^*_\alpha - 2}}}{(2-r)^{\frac{r+2-22^*_\alpha}{22^*_\alpha - 2}}} \|(u,v)\|^{\frac{2(22^*_\alpha - r - 2)}{22^*_\alpha - 2}} (Q(u,v))^{\frac{r}{22^*_\alpha - 2}} \\
&< 0.
\end{aligned}$$

Thus, $\mathcal{S}_{u,v}(t)$ has maximum value at $t_{\max}$. Moreover, we have

$$\Psi'_{u,v}(t) = t^r \left(\mathcal{S}_{u,v}(t) - P_{\lambda,\mu}(u,v) \right). \tag{3.3.22}$$

Lemma 3.3.4. *Assume that* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$ *and* $(u,v) \in \mathcal{H}$, *the following results hold:*

(i) If $Q(u,v) < 0$ *and* $P_{\lambda,\mu}(u,v) < 0$, *then there does not exists any critical point.*

(ii) If $Q(u,v) \leq 0$ *and* $P_{\lambda,\mu}(u,v) < 0$, *then there exists a unique* (t^+u, t^+v) *such that* $(t^+u, t^+v) \in \mathcal{N}^+_{\lambda,\mu}$ *and* $I_{\lambda,\mu}(t^+u, t^+v) = \inf_{t \geq 0} I_{\lambda,\mu}(tu, tv)$.

(iii) If $Q(u,v) > 0$ *and* $P_{\lambda,\mu}(u,v) \leq 0$, *then there exists a unique* $t^- > t_{\max}$ *such*

that $(t^-u, t^-v) \in \mathcal{N}_{\lambda,\mu}^-$ *and* $I_{\lambda,\mu}(t^-u, t^-v) = \sup_{t \geq t_{\max}} I_{\lambda,\mu}(tu, tv)$.

(*iv*) *If* $Q(u,v) > 0$ *and* $P_{\lambda,\mu}(u,v) > 0$*, then there exists unique* t^+ *and* t^- *satisfying* $0 < t^+ < t_{\max} < t^-$ *such that* $(t^+u, t^+v) \in \mathcal{N}_{\lambda,\mu}^+$ *and* $(t^-u, t^-v) \in \mathcal{N}_{\lambda,\mu}^-$. *Moreover*

$$I_{\lambda,\mu}(t^+u, t^+v) = \inf_{0 \leq t \leq t_{\max}} I_{\lambda,\mu}(tu, tv);\ I_{\lambda,\mu}(t^-u, t^-v) = \sup_{t \geq t_{\max}} I_{\lambda,\mu}(tu, tv).$$

Proof. Let $(0,0) \neq (u,v) \in \mathcal{H}$, then we have following four possible cases:

(*i*) If $Q(u,v) < 0$ and $P_{\lambda,\mu}(u,v) < 0$, then $\Psi_{u,v}(t) = 0$ at $t = 0$ and $\Psi'_{u,v}(t) > 0$ for all $t > 0$. This implies that Φ_u is strictly increasing and hence no critical point.

(*ii*) If $Q(u,v) < 0$, then from (3.3.21) $\mathcal{S}_{u,v}$ is strictly increasing for $t > 0$. As $P_{\lambda,\mu}(u,v) \geq 0$, this implies that there exists a unique t^+ such that $\mathcal{S}_{u,v}(t^+) = P_{\lambda,\mu}(u,v)$ with $\mathcal{S}_{u,v}(t^+) > 0$. Using (3.3.22), we conclude that $(t^+u, t^+v) \in \mathcal{N}_{\lambda,\mu}$. Further, $\Psi'_{u,v}(t) > 0$ and $\Psi'_{u,v}(t) < 0$ for $t > t^+$ and and $t < t^+$ respectively. Also $\Psi''_{u,v}(t^+) = (t^+)^{1+r}\mathcal{S}'_{u,v}(t^+) > 0$. Thus, $(t^+u, t^+v) \in \mathcal{N}_{\lambda,\mu}^+$ and $I_{\lambda,\mu}(t^+u, t^+v) = \inf_{t \geq 0} I_{\lambda,\mu}(tu, tv)$.

(*iii*) If $Q(u,v) > 0$, then $t_{\max}$ is the point at which $\mathcal{S}'_{u,v}(t) > 0$ has maximum. Thus, $\mathcal{S}_{u,v}(t)$ is strictly increasing for $0 \leq t < t_{\max}$ and strictly decreasing for $t_{\max} < t < \infty$. As $P_{\lambda,\mu}(u,v) \leq 0$, so there is a unique $t^- > t_{\max} > 0$ such that $\mathcal{S}_{u,v}(t^-) = P_{\lambda,\mu}(u,v)$ and $\mathcal{S}_{u,v}(t^-) < 0$. Further, (3.3.22) gives $\Psi'_{u,v}(t^-) = 0$. Thus $(t^-u, t^-v) \in \mathcal{N}_{\lambda,\mu}$. Also, $\Psi''_{u,v}(t^-) = (t^-)^{1+r}\mathcal{S}'_{u,v}(t^-) < 0$ and $\Psi'_{u,v}(t) < 0$ for $t > t_{\max}$, so $\Psi_{u,v}(t^-) = \sup_{t \geq t_{\max}} \Psi_{u,v}(t)$. Hence, $(t^-u, t^-v) \in \mathcal{N}_{\lambda,\mu}^-$ and $I_{\lambda,\mu}(t^-u, t^-v) = \sup_{t \geq t_{\max}} I_{\lambda,\mu}(tu, tv)$.

(*iv*) Since $Q(u,v) > 0$, $\mathcal{S}_{u,v}(t)$ achieves its maximum at $t = t_{\max}$. Thus

$$
\begin{aligned}
\mathcal{S}_{u,v}(t_{\max}) =& \|(u,v)\|^r \left(\frac{2-r}{2(22^*_\alpha - r)}\right)^{\frac{2-r}{22^*_\alpha-2}} \left(\frac{22^*_\alpha - 2}{22^*_\alpha - r}\right) \left(\frac{\|(u,v)\|^{22^*_\alpha}}{Q(u,v)}\right)^{\frac{2-r}{22^*_\alpha-2}} \\
\geq& \left[\left(\frac{2-r}{2(22^*_\alpha - r)}\right) \|H^+\|_\infty^{-2} (\bar{S}_{H,L})^{2^*_\alpha}\right]^{\frac{2-r}{22^*_\alpha-2}} \left(\frac{22^*_\alpha - 2}{22^*_\alpha - r}\right) \|(u,v)\|^r.
\end{aligned}
$$

As $P_{\lambda,\mu}(u,v) > 0$, so

$$
\begin{aligned}
\mathcal{S}_{u,v}(t_{\max}) - P_{\lambda,\mu}(u,v) \geq& \left[\frac{2-r}{2(22^*_\alpha - r)} \frac{(\bar{S}_{H,L})^{2^*_\alpha}}{\|H^+\|_\infty^2}\right]^{\frac{2-r}{22^*_\alpha-2}} \left(\frac{22^*_\alpha - 2}{22^*_\alpha - r}\right) \|(u,v)\|^r \\
&- \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}} S^{-\frac{r}{2}} \|(u,v)\|^r \\
>& \, 0,
\end{aligned}
$$

for $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$. Thus, there exists t^+ and t^- with $0 < t^+ < t_{\max} < t^-$ satisfying,

$$
\mathcal{S}_{u,v}(t^+) = P_{\lambda,\mu}(u,v) = \mathcal{S}_{u,v}(t^-) \text{ and } \mathcal{S}'_{u,v}(t^+) < 0 < \mathcal{S}'_{u,v}(t^-).
$$

Therefore, $(t^+u, t^+v) \in \mathcal{N}^+_{\lambda,\mu}$, $(t^-u, t^-v) \in \mathcal{N}^-_{\lambda,\mu}$. Furthermore, $\Psi'_{u,v}(t) < 0$ for $t \in (0, t^+)$, $\Psi'_u(t) > 0$ for $t \in (t^+, t^-)$ and $\Psi'_{u,v}(t) < 0$ for $t \in (t^-, \infty)$.
Hence

$$
I_{\lambda,\mu}(t^+u, t^+v) = \inf_{0 \leq t \leq t_{\max}} I_{\lambda,\mu}(tu, tv); \; I_{\lambda,\mu}(t^-u, t^-v) = \sup_{t \geq t_{\max}} I_{\lambda,\mu}(tu, tv).
$$

This completes the proof. □

Lemma 3.3.5. *If* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$, *then* $\mathcal{N}^0_{\lambda,\mu}$ *is a null set.*

Proof. We will prove it by contradiction. Let $(u,v) \in \mathcal{N}^0_{\lambda,\mu}$, then (3.3.20) implies that,

$$
\|(u,v)\|^2 = \frac{22^*_\alpha - r}{22^*_\alpha - 2} P_{\lambda,\mu}(u,v) \tag{3.3.23}
$$

and

$$\|(u,v)\|^2 = \frac{2(22^*_\alpha - r)}{2-r} Q(u,v). \tag{3.3.24}$$

On using (3.1.6) in (3.3.23), it is easy to calculate

$$\|(u,v)\| \leq \left(\frac{22^*_\alpha - r}{22^*_\alpha - 2} S^{-\frac{r}{2}} \right)^{\frac{1}{2-r}} \left((\lambda \|F\|_\beta)^{\frac{2}{2-r}} + (\mu \|G\|_\beta)^{\frac{2}{2-r}} \right)^{\frac{1}{2}}. \tag{3.3.25}$$

Now, subtituting (3.1.7) in (3.3.24), we obtain

$$\|(u,v)\| \geq \left[\left(\frac{2-r}{2(22^*_\alpha - r)} \right) \|H^+\|_\infty^{-2} (\bar{S}_{H,L})^{2^*_\alpha} \right]^{\frac{1}{22^*_\alpha - 2}}. \tag{3.3.26}$$

Thus, from (3.3.25) and (3.3.26), we get

$$(\lambda \|F\|_\beta)^{\frac{2}{2-r}} + (\mu \|G\|_\beta)^{\frac{2}{2-r}} \geq \left(\frac{22^*_\alpha - 2}{22^*_\alpha - r} \right)^{\frac{2}{2-r}} \left[\frac{2-r}{2(22^*_\alpha - r)} \frac{(\bar{S}_{H,L})^{2^*_\alpha}}{\|H^+\|_\infty^2} \right]^{\frac{1}{2^*_\alpha - 1}} S^{\frac{r}{2-r}} := \Upsilon_1,$$

which contradicts the fact that $0 < (\lambda \|F\|_\beta)^{\frac{2}{2-r}} + (\mu \|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$. Hence $\mathcal{N}^0_{\lambda,\mu} = \phi$. □

Consequently, if $0 < (\lambda \|F\|_\beta)^{\frac{2}{2-r}} + (\mu \|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$, then we have

$$\mathcal{N}_{\lambda,\mu} = \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu}.$$

Now, we define

$$k_{\lambda,\mu} := \inf_{(u,v)\in \mathcal{N}_{\lambda,\mu}} I_{\lambda,\mu}(u,v) \ ; \ k^+_{\lambda,\mu} := \inf_{(u,v)\in \mathcal{N}^+_{\lambda,\mu}} I_{\lambda,\mu}(u,v) \ ; \ k^-_{\lambda,\mu} := \inf_{(u,v)\in \mathcal{N}^-_{\lambda,\mu}} I_{\lambda,\mu}(u,v).$$

Lemma 3.3.6. *The following facts hold:*

(i) *If* $0 < (\lambda \|F\|_\beta)^{\frac{2}{2-r}} + (\mu \|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$, *then* $k_{\lambda,\mu} \leq k^+_{\lambda,\mu} < 0$.

(ii) *If* $0 < \lambda < \frac{r}{2}\Upsilon_1$, *then* $k^-_{\lambda,\mu} > d_0$, *where* d_0 *is a positive constant depending on* $\lambda, \mu, \alpha, r, N$, S, $\|F\|_\alpha$, $\|G\|_\alpha$ *and* $\|H^+\|_\infty$.

Proof. (i) Let $(u,v) \in \mathcal{N}^{+}_{\lambda,\mu}$, then (3.3.20) gives

$$\frac{2-r}{2(22^*_\alpha - r)}\|(u,v)\|^2 > Q(u,v).$$

This together with (3.1.5) and (3.3.18) yield

$$I_{\lambda,\mu}(u,v) = \left(\frac{1}{2}-\frac{1}{r}\right)\|(u,v)\|^2 + 2\left(\frac{1}{r}-\frac{1}{22^*_\alpha}\right)Q(u,v) < -\frac{(2-r)(2^*_\alpha - 1)}{r22^*_\alpha}\|(u,v)\|^2 < 0.$$

Thus, by the definition of $k_{\lambda,\mu}$ and $k^{+}_{\lambda,\mu}$, we conclude that $k_{\lambda,\mu} \leq k^{+}_{\lambda,\mu} < 0$.
(ii) Let $(u,v) \in \mathcal{N}^{-}_{\lambda,\mu}$. Then using (3.3.20) and (3.1.7), we have

$$\frac{2-r}{2(22^*_\alpha - r)}\|(u,v)\|^2 < Q(u,v) \leq \|H^+\|^2_\infty \left(\bar{S}_{H,L}\right)^{-2^*_\alpha}\|(u,v)\|^{22^*_\alpha}.$$

This implies that

$$\|(u,v)\| > \left(\frac{2-r}{2(22^*_\alpha - r)}\|H^+\|^{-2}_\infty \left(\bar{S}_{H,L}\right)^{2^*_\alpha}\right)^{\frac{1}{22^*_\alpha - 2}}. \tag{3.3.27}$$

On combining (3.3.19) and (3.3.27), we obtain

$$\begin{aligned} I_{\lambda,\mu}(u,v) \geq &\|(u,v)\|^r \left[\left(\frac{1}{2}-\frac{1}{22^*_\alpha}\right)\|(u,v)\|^{2-r} - \left(\frac{1}{r}-\frac{1}{22^*_\alpha}\right)S^{-\frac{r}{2}} \right. \\ &\left. \times \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}}\right] \\ > &\left(\frac{2-r}{2(22^*_\alpha - r)}\frac{\left(\bar{S}_{H,L}\right)^{2^*_\alpha}}{\|H^+\|^2_\infty}\right)^{\frac{r}{22^*_\alpha - 2}} \left[\left(\frac{1}{2}-\frac{1}{22^*_\alpha}\right)\left(\frac{2-r}{2(22^*_\alpha - r)}\frac{\left(\bar{S}_{H,L}\right)^{2^*_\alpha}}{\|H^+\|^2_\infty}\right)^{\frac{2-r}{22^*_\alpha - r}} \right. \\ &\left. - \left(\frac{1}{r}-\frac{1}{22^*_\alpha}\right)S^{-\frac{r}{2}}\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}}\right] \end{aligned}$$

Thus, if $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \left(\frac{r}{2}\right)^{\frac{2}{2-r}}\Upsilon_1$, then $I_{\lambda,\mu}(u,v) > d_0$ for all $(u,v) \in \mathcal{N}^{-}_{\lambda,\mu}$, where d_0 is a positive constant depending on $\lambda, \mu, \alpha, r, N$, S, $\|F\|_\alpha$, $\|G\|_\alpha$ and $\|H^+\|_\infty$. □

3.4 Existence of the first local minimizer

In this section, firstly we show the existence of Palais-Smale sequence corresponding to energy functional $I_{\lambda,\mu}$ in $\mathcal{N}^{\pm}_{\lambda,\mu}$, by using the implicit function theorem.

Lemma 3.4.1. *Suppose* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$. *Then for every* $z = (u,v) \in \mathcal{N}_{\lambda,\mu}$, *there exist* $\epsilon > 0$ *and a differentiable mapping* $\zeta : B(0,\epsilon) \subset \mathcal{H} \to \mathbb{R}^+$ *such that* $\zeta(0) = 1$, $\zeta(w)(z-w) \in \mathcal{N}_{\lambda,\mu}$ *and for all* $w = (w_1, w_2) \in \mathcal{H}$

$$\langle \zeta'(0), w\rangle = \frac{2\mathcal{B}(z,w) - r\mathcal{P}_{\lambda,\mu}(z,w) - 2\mathcal{Q}(z,w)}{(2-r)\|(u,v)\|^2 - 2(22^*_\alpha - r)Q(u,v)},$$

where

$$\begin{aligned}
\mathcal{B}(z,w) &= \int_\Omega (\Delta u \Delta w_1 + \Delta v \Delta w_2)\, dx,\\
\mathcal{P}_{\lambda,\mu}(z,w) &= \int_\Omega \left(\lambda F(x)|u|^{r-2}uw_1 + \mu G(x)|v|^{r-2}vw_2\right) dx,\\
\mathcal{Q}(z,w) &= \int_\Omega\int_\Omega H(x)H(y)\left(\frac{|v(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha-2}u(y)z_1 + |u(x)|^{2^*_\alpha}|v(y)|^{2^*_\alpha-2}v(y)z_2}{|x-y|^\alpha}\right)
\end{aligned}$$

Proof. For $z = (u,v) \in \mathcal{N}_{\lambda,\mu}$, define a map $\xi_z : \mathbb{R} \times \mathcal{H} \to \mathbb{R}$ such that

$$\begin{aligned}
\xi_z(\zeta, w) = \langle I'_{\lambda,\mu}(\zeta(z-w)), \zeta(z-w)\rangle &= \zeta^2\|(u-w_1, v-w_2)\|^2\\
&- \zeta^r \int_\Omega (\lambda F(x)|u-w_1|^r + \mu G(x)|v-w_2|^r)dx - 2\zeta^{22^*_\alpha} Q(u-w_1, v-w_2)dx
\end{aligned}$$

Then $\xi_z(1,(0,0)) = \langle I'_{\lambda,\mu}(z), z\rangle = 0$ and

$$\begin{aligned}
\frac{d}{d\zeta}\xi_z(1,(0,0)) =& 2\|(u,v)\|^2 - r\int_\Omega (\lambda F(x)|u|^r + \mu G(x)|v|^r)dx - 2(22^*_\alpha)Q(u,v)\\
=& (2-r)\|(u,v)\|^2 - 2(22^*_\alpha - r)Q(u,v) \neq 0.
\end{aligned}$$

Further, by the same argument used in Lemma 2.3.7 of chapter 2, we get the required result. □

The similar result is also true for $(u,v) \in \mathcal{N}^-_{\lambda,\mu}$, which is as follows.

Lemma 3.4.2. *Suppose* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$. *Then for every* $z = (u,v) \in \mathcal{N}^-_{\lambda,\mu}$, *there exist* $\epsilon > 0$ *and a differentiable mapping* $\zeta^- : B(0,\epsilon) \subset \mathcal{H} \to \mathbb{R}^+$ *such that* $\zeta^-(0) = 1$, $\zeta^-(w)(z-w) \in \mathcal{N}^-_{\lambda,\mu}$ *and for all* $w = (w_1, w_2) \in \mathcal{H}$

$$\langle (\zeta^-)'(0), w\rangle = \frac{2\mathcal{B}(z,w) - r\mathcal{P}_{\lambda,\mu}(z,w) - 2\mathcal{Q}(z,w)}{(2-r)\|(u,v)\|^2 - 2(22^*_\alpha - r)Q(u,v)},$$

where $\mathcal{B}(z,w)$, $\mathcal{P}_{\lambda,\mu}(z,w)$ *and* $\mathcal{Q}(z,w)$ *is same as in Lemma 3.4.1.*

Proof. The proof follows similarly as done in Lemma 3.4.1. □

Lemma 3.4.3. *The following statements are true:*

(*i*) *If* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$, *then there exists a* $(PS)_{k_{\lambda,\mu}}$*-sequence* $\{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\mu}$ *in* $\mathcal{H}$ *for* $I_{\lambda,\mu}$.

(*ii*) *If* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \left(\frac{r}{2}\right)^{\frac{2}{2-r}} \Upsilon_1$, *then there exists a* $(PS)_{k^-_{\lambda,\mu}}$*-sequence* $\{(u_n, v_n)\} \subset \mathcal{N}^-_{\lambda,\mu}$ *in* $\mathcal{H}$ *for* $I_{\lambda,\mu}$.

Proof. (*i*) According to Lemma 3.3.1 and Ekeland Variational Principle [27], there exists a minimizing sequence $\{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\mu}$ such that

$$I_{\lambda,\mu}(u_n, v_n) < k_{\lambda,\mu} + \frac{1}{n},$$
$$I_{\lambda,\mu}(u_n, v_n) < I_{\lambda,\mu}(u,v) + \frac{1}{n}\|(u,v) - (u_n, v_n)\|, \text{ for each } (u,v) \in \mathcal{N}_{\lambda,\mu}.$$

Using Lemma 3.3.6(*i*) and taking n large, we get

$$\begin{aligned} I_{\lambda,\mu}(u_n, v_n) &= \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right)\|(u_n, v_n)\|^2 - \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right) P_{\lambda,\mu}(u_n, v_n) \\ &< k_{\lambda,\mu} + \frac{1}{n} < \frac{k_{\lambda,\mu}}{2}. \end{aligned} \tag{3.4.28}$$

This implies that

$$0 < -\frac{r2^*_\alpha k_{\lambda,\mu}}{22^*_\alpha - r} < P_{\lambda,\mu}(u_n, v_n) \le S^{-\frac{r}{2}} \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}} \|(u_n, v_n)\|^r. \tag{3.4.29}$$

Therefore, $(u_n, v_n) \neq (0,0)$. Now, from (3.4.28), we have

$$\|(u_n, v_n)\| \leq \left[\left(\frac{22^*_\alpha - r}{r(2^*_\alpha - 1)} \right) S^{-\frac{r}{2}} \left((\lambda \|F\|_\beta)^{\frac{2}{2-r}} + (\mu \|G\|_\beta)^{\frac{2}{2-r}} \right)^{\frac{2-r}{2}} \right]^{\frac{1}{2-r}}. \quad (3.4.30)$$

Further, (3.4.29) gives us

$$\|(u_n, v_n)\| \geq \left[-\frac{r2^*_\alpha k_{\lambda,\mu}}{22^*_\alpha - r} S^{\frac{r}{2}} \left((\lambda \|F\|_\beta)^{\frac{2}{2-r}} + (\mu \|G\|_\beta)^{\frac{2}{2-r}} \right)^{\frac{r-2}{2}} \right]^{\frac{1}{r}}.$$

Further, using the same idea of the proof as done in Lemma 2.3.9 of chapter 2, one can easily show that $\|I'_{\lambda,\mu}(u_n, v_n)\|_{\mathcal{H}^{-1}} \to 0, \text{ as } n \to \infty$. Also, we show that $\|\zeta'_n(0)\|$ is uniformly bounded.

By Hölder's inequality and Sobolev embedding theorem, we have

$$\begin{aligned}
&\int_\Omega \lambda F(x)|u_n|^{r-1} w_1 dx + \int_\Omega \mu G(x)|v_n|^{r-1} w_2 dx \\
&\quad \leq \lambda \|F\|_\alpha \left(\int_\Omega \left(|u_n|^{r-1} w_1 \right)^{\frac{2^*}{r}} \right)^{\frac{r}{2^*}} + \mu \|G\|_\alpha \left(\int_\Omega \left(|v_n|^{r-1} w_2 \right)^{\frac{2^*}{r}} \right)^{\frac{r}{2^*}} \\
&\quad \leq \lambda \|F\|_\alpha \|u_n\|_{2^*}^{r-1} \|w_1\|_{2^*} + \mu \|G\|_\alpha \|v_n\|_{2^*}^{r-1} \|w_2\|_{2^*} \\
&\quad \leq S^{-\frac{r}{2}} \left(\lambda \|F\|_\alpha + \mu \|G\|_\alpha \right) \|(u_n, v_n)\|^{r-1} \|(w_1, w_2)\|. \qquad (3.4.31)
\end{aligned}$$

Using Hardy-Littlewood-Sobolev inequality, Hölder's inequality and Sobolev embedding theorem, we obtain

$$\begin{aligned}
&\int_\Omega \int_\Omega \left(\frac{|u_n|^{2^*_\alpha}}{|x-y|^\alpha} \right) |v_n|^{2^*_\alpha - 1} w_1 dx dy \\
&\leq C(N, \alpha) \left(\int_\Omega |u_n|^{\frac{22^*_\alpha N}{2N-\alpha}} \right)^{\frac{2N-\alpha}{2N}} \left(\int_\Omega \left(|v_n|^{2^*_\alpha - 1} w_1 \right)^{\frac{2N}{2N-\alpha}} \right)^{\frac{2N-\alpha}{2N}} \\
&\leq C(N, \alpha) \left(\int_\Omega |u_n|^{2^*} \right)^{\frac{2N-\alpha}{2N}} \left(\int_\Omega |v_n|^{2^*} \right)^{\frac{N+4-\alpha}{2N-\alpha}} \left(\int_\Omega |w_1|^{2^*} \right)^{\frac{1}{2^*}} \\
&\leq \left[\left(S^{-1} \int_\Omega |\Delta u_n|^2 \right)^{\frac{2^*}{2}} \right]^{\frac{2N-\alpha}{2N}} \left[\left(S^{-1} \int_\Omega |\Delta v_n|^2 \right)^{\frac{2^*}{2}} \right]^{\frac{N+4-\alpha}{2N}} \left[\left(S^{-1} \int_\Omega |\Delta w_1|^2 \right)^{\frac{2^*}{2}} \right]^{\frac{1}{2^*}} \\
&\leq A_2 \|u_n\|^{2^*_\alpha} \|v_n\|^{\frac{N+4-\alpha}{N-4}} \|w_1\| \leq A_2 \|(u_n, v_n)\|^{\frac{3N+4-2\alpha}{N-4}} \|(w_1, w_2)\|.
\end{aligned}$$

Using the same idea, we can calculate

$$\int_\Omega \int_\Omega \left(\frac{|v_n|^{2^*_\alpha}}{|x-y|^\alpha} \right) |u_n|^{2^*_\alpha - 1} w_2 dxdy \leq A_3 \|(u_n, v_n)\|^{\frac{3N+4-2\alpha}{N-4}} \|(w_1, w_2)\|. \qquad (3.4.32)$$

Thus, on combining (3.4.31)-(3.4.32) and (3.4.30), we have

$$|(\zeta_n'(0), w)| \leq \frac{A_4 \|(w_1, w_2)\|}{|(2-r)\|(u_n, v_n)\|^2 - 2(22^*_\alpha - r) Q(u_n, v_n)|},$$

where $A_4 > 0$ is a constant. Now we are left to show that

$$\left|(2-r)\|(u_n, v_n)\|^2 - 2(22^*_\alpha - r) Q(u_n, v_n)\right| \geq A_5,$$

for some $A_5 > 0$ and n is taking large enough. On contradiction argue, suppose there exists a subsequence $\{(u_n, v_n)\}$ such that

$$\left|(2-r)\|(u_n, v_n)\|^2 - 2(22^*_\alpha - r) Q(u_n, v_n)\right| = o_n(1). \qquad (3.4.33)$$

From (3.4.33) and using $(u_n, v_n) \in \mathcal{N}_{\lambda,\mu}$, we have

$$\|(u_n, v_n)\|^2 = \frac{22^*_\alpha - r}{22^*_\alpha - 2} P_{\lambda,\mu}(u_n, v_n) + o_n(1),$$

and

$$\|(u_n, v_n)\|^2 = \frac{2(22^*_\alpha - r)}{2-r} Q(u_n, v_n) + o_n(1).$$

Then Hölder's inequality, Sobolev embedding theorem, and definition of $\bar{S}_{H,L}$ yield

$$\|(u_n, v_n)\| \leq \left(\frac{22^*_\alpha - r}{22^*_\alpha - 2} S^{-\frac{r}{2}} \right)^{\frac{1}{2-r}} \left((\lambda \|F\|_\beta)^{\frac{2}{2-r}} + (\mu \|G\|_\beta)^{\frac{2}{2-r}} \right)^{\frac{1}{2}} + o_n(1).$$
$$\|(u_n, v_n)\| \geq \left[\left(\frac{2-r}{2(22^*_\alpha - r)} \right) \|H^+\|_\infty^{-2} (2S_{H,L})^{2^*_\alpha} \right]^{\frac{1}{22^*_\alpha - 2}} + o_n(1).$$

This implies that $(\lambda \|F\|_\beta)^{\frac{2}{2-r}} + (\mu \|G\|_\beta)^{\frac{2}{2-r}} \geq \Upsilon_1$, which is a contradiction to the fact that $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$. Hence,

$$\left\langle I'_{\lambda,\mu}(u_n, v_n), \frac{(u,v)}{\|(u,v)\|} \right\rangle \leq \frac{A_1}{n}.$$

Thus, proof of (i) is completed.

(ii) Using Lemma 3.4.2, one can prove (ii) in a similar manner. □

Lemma 3.4.4. *Let* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$, *then* $I_{\lambda,\mu}$ *has a minimizer* $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ *in* $\mathcal{N}^+_{\lambda,\mu}$ *which satisfies the following:*

- (i) $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = k_{\lambda,\mu} = k^+_{\lambda,\mu} < 0$.
- (ii) $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ *is a nontrivial solution of the system* $(\mathcal{D}_{\lambda,\mu})$.
- (iii) $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \to (0,0)$ *as* $\lambda \to 0^+$, $\mu \to 0^+$.

Proof. By Lemma 3.4.3 (i), there exists a minimizing sequence $\{(u_n, v_n)\}$ for $I_{\lambda,\mu}$ such that

$$I_{\lambda,\mu}(u_n, v_n) = k_{\lambda,\mu} + o_n(1),\ I'_{\lambda,\mu}(u_n, v_n) = o_n(1) \text{ in } \mathcal{H}^{-1}.$$

Lemma 3.4.1 gives us that $\{(u_n, v_n)\}$ is bounded in $\mathcal{H}$. So up to subsequence $(u_n, v_n) \rightharpoonup (u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ weakly in $(u_n, v_n) \to \mathcal{H}$, $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ strongly in $L^m(\Omega)$ $\forall 1 \leq m < 2^*$ and $(u_n(x), v_n(x)) \to (u^1_{\lambda,\mu}(x), v^1_{\lambda,\mu}(x))$ pointwise a.e. in Ω. Then, it is easy to see that

$$\begin{aligned} |u_n|^{2^*_\alpha} \rightharpoonup |u^1_{\lambda,\mu}|^{2^*_\alpha}, |v_n|^{2^*_\alpha} \rightharpoonup |v^1_{\lambda,\mu}|^{2^*_\alpha} \text{ in } L^{\frac{2N}{2N-\alpha}}(\Omega) \text{ and} \\ |u_n|^{2^*_\alpha-2}u_n \rightharpoonup |u^1_{\lambda,\mu}|^{2^*_\alpha-2}u^1_{\lambda,\mu}, |v_n|^{2^*_\alpha-2}v_n \rightharpoonup |v^1_{\lambda,\mu}|^{2^*_\alpha-2}v^1_{\lambda,\mu} \text{ in } L^{\frac{2N}{N+4-\alpha}}(\Omega), \end{aligned} \tag{3.4.34}$$

as $n \to \infty$. As we know that the Riesz potential defines a continuous linear map from $L^{\frac{2N}{2N-\alpha}}(\Omega)$ to $L^{\frac{2N}{\alpha}}(\Omega)$ which provides

$$\left.\begin{aligned} |x|^{-\alpha} * |u_n|^{2^*_\alpha} \rightharpoonup |x|^{-\alpha} * |u^1_{\lambda,\mu}|^{2^*_\alpha}, \\ |x|^{-\alpha} * |v_n|^{2^*_\alpha} \rightharpoonup |x|^{-\alpha} * |v^1_{\lambda,\mu}|^{2^*_\alpha}, \end{aligned}\right\} \text{ weakly in } L^{\frac{2N}{\alpha}}(\Omega), \tag{3.4.35}$$

as $n \to \infty$. Thus, (3.4.34) and (3.4.35) gives us

$$\left.\begin{array}{ll}\left(|x|^{-\alpha} * |v_n|^{2^*_\alpha}\right)|u_n|^{2^*_\alpha-2}u_n & \rightharpoonup \left(|x|^{-\alpha} * |v^1_{\lambda,\mu}|^{2^*_\alpha}\right)|u^1_{\lambda,\mu}|^{2^*_\alpha-2}u^1_{\lambda,\mu},\\ \left(|x|^{-\alpha} * |u_n|^{2^*_\alpha}\right)|v_n|^{2^*_\alpha-2}v_n & \rightharpoonup \left(|x|^{-\alpha} * |u^1_{\lambda,\mu}|^{2^*_\alpha}\right)|v^1_{\lambda,\mu}|^{2^*_\alpha-2}v^1_{\lambda,\mu},\end{array}\right\}, \tag{3.4.36}$$

weakly in $L^{\frac{2N}{N+4}}(\Omega)$ as $n \to \infty$. Therefore, for any $(\phi, \psi) \in \mathcal{H}$, we have

$$\begin{aligned}\lim_{n\to\infty}\Bigg[\int_\Omega (\Delta u_n \Delta\phi + \Delta v_n \Delta\psi)\,dx - \int_\Omega \left(\lambda F(x)|u_n|^{r-2}u\phi + \mu G(x)|v_n|^{r-2}v\psi\right)dx\\ - \int_\Omega\int_\Omega H(x)H(y)\frac{|v_n(x)|^{2^*_\alpha}|u_n(y)|^{2^*_\alpha-2}u_n(y)\phi(y)}{|x-y|^\alpha}dxdy\\ - \int_\Omega\int_\Omega H(x)H(y)\frac{|u_n(x)|^{2^*_\alpha}|v_n(y)|^{2^*_\alpha-2}v_n(y)\psi(y)}{|x-y|^\alpha}dxdy\Bigg] = 0,\end{aligned}$$

because of $\|I'_{\lambda,\mu}(u_n, v_n)\| \to 0$ as $n \to \infty$. Thus, using (3.4.36), continuity of H and passing the limit as $n \to \infty$, we have

$$\begin{aligned}\int_\Omega \left(\Delta u^1_{\lambda,\mu}\Delta\phi + \Delta v^1_{\lambda,\mu}\Delta\psi\right)dx - \int_\Omega \left(\lambda F(x)|u^1_{\lambda,\mu}|^{r-2}u^1_{\lambda,\mu}\phi + \mu G(x)|v^1_{\lambda,\mu}|^{r-2}v^1_{\lambda,\mu}\psi\right)dx\\ - \int_\Omega\int_\Omega H(x)H(y)\frac{|v^1_{\lambda,\mu}(x)|^{2^*_\alpha}|u^1_{\lambda,\mu}(y)|^{2^*_\alpha-2}u^1_{\lambda,\mu}(y)\phi(y)}{|x-y|^\alpha}dxdy\\ - \int_\Omega\int_\Omega H(x)H(y)\frac{|u^1_{\lambda,\mu}(x)|^{2^*_\alpha}|v^1_{\lambda,\mu}(y)|^{2^*_\alpha-2}v^1_{\lambda,\mu}(y)\psi(y)}{|x-y|^\alpha}dxdy = 0,\end{aligned}$$

i.e. $\left\langle I'_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}), (\phi, \psi)\right\rangle \to 0$. This implies that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is a weak solution of $(\mathcal{D}_{\lambda,\mu})$. Since $(u_n, v_n) \in \mathcal{N}_{\lambda,\mu}$. So, we have

$$\|(u_n, v_n)\|^2 = P_{\lambda,\mu}(u_n, v_n) + 2Q(u_n, v_n),$$

which gives

$$\begin{aligned}I_{\lambda,\mu}(u_n, v_n) &= \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right)\|(u_n, v_n)\|^2 - \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right)P_{\lambda,\mu}(u_n, v_n)\\ &\geq -\left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right)P_{\lambda,\mu}(u_n, v_n).\end{aligned}$$

Taking $n \to \infty$ together with $\lambda,\ \mu < 0$, we obtain

$$P_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \geq -\frac{22^*_\alpha k_{\lambda,\mu}}{(22^*_\alpha - r)} > 0. \tag{3.4.37}$$

Therefore, $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is a nontrivial solution of $(\mathcal{D}_{\lambda,\mu})$. Afterwards, we will show that $(u_n, v_n) \to (u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ strongly in $\mathcal{H}$ and $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = k_{\lambda,\mu}$. Using Fatou's Lemma, we obtain

$$\begin{aligned} k_{\lambda,\mu} \leq I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) &= \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right) \|(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})\|^2 - \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right) P_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \\ &\leq \liminf_{n\to\infty} \left[\left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right) \|(u_n, v_n)\|^2 - \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right) P_{\lambda,\mu}(u_n, v_n)\right] \\ &= \liminf_{n\to\infty} I_{\lambda,\mu}(u_n, v_n) = k_{\lambda,\mu}. \end{aligned}$$

This implies that $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = k_{\lambda,\mu}$ and $\lim\limits_{n\to\infty} \|(u_n, v_n)\|^2 = \|(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})\|^2$. Further, Brézis and Lieb Lemma [12] contributes that $(u_n, v_n) \to (u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ strongly in $\mathcal{H}$.

Now, we are left to show that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$. We prove this by contradiction argument. Suppose $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$ Then, from Lemma 3.3.3 (ii) and (3.4.37), we have

$$Q(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) > 0 \text{ and } P_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) > 0.$$

Thus, from Lemma 3.3.4, there exist unique t_1^+ and t_1^- such that $(t_1^+ u^1_{\lambda,\mu}, t_1^+ v^1_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$ and $(t_1^- u^1_{\lambda,\mu}, t_1^- v^1_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$. In particular, we have $t_1^+ < t_1^- = 1$. Since $\Psi'_{(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})}(t_1^+) = 0$ and $\Psi''_{(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})}(t_1^+) > 0$, there exists $t_1^+ < \bar{t} \leq t_1^-$ such that $I_{\lambda,\mu}(t_1^+ u^1_{\lambda,\mu}, t_1^+ v^1_{\lambda,\mu}) < I_{\lambda,\mu}(\bar{t} u^1_{\lambda,\mu}, \bar{t} v^1_{\lambda,\mu})$. On using Lemma 3.3.4, we obtain

$$I_{\lambda,\mu}(t_1^+ u^1_{\lambda,\mu}, t_1^+ v^1_{\lambda,\mu}) < I_{\lambda,\mu}(\bar{t} u^1_{\lambda,\mu}, \bar{t} v^1_{\lambda,\mu}) \leq I_{\lambda,\mu}(t_1^- u^1_{\lambda,\mu}, t_1^- v^1_{\lambda,\mu}) = I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = k_{\lambda,\mu},$$

which is a contradiction. Therefore, $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$.

(iii) Further, from Lemma 3.3.6 (i) and (3.3.19), we have

$$0 > k^+_{\lambda,\mu} \geq k_{\lambda,\mu} = I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) > -\left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right) S^{-\frac{r}{2}} \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}} \times \|(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})\|^r,$$

which implies that $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \to (0,0)$ as $\lambda \to 0^+$, $\mu \to 0^+$ which completes the proof. □

Proof of Theorem 3.0.1 From Lemma 3.4.4, we conclude that $(\mathcal{D}_{\lambda,\mu})$ has a non-trivial solution $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$. □

3.5 Existence of a second local minimizer

In this segment, we first prove the critical level by using few estimates which are already proved in section 3.1. Then we show the existence of a second weak solution of problem $(\mathcal{D}_{\lambda,\mu})$ under the assumptions $(Z1) - (Z4)$. At the end of this section, we give the proof of Theorem 3.0.2.

Lemma 3.5.1. *Assume that $(Z1) - (Z4)$ hold and $\frac{N}{N-4} \leq r < 2$, then there exist $(u_{\lambda,\mu}, v_{\lambda,\mu})$ in $\mathcal{H}\backslash\{(0,0)\}$ and $\Upsilon > 0$ such that for $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon$,*

$$\sup_{t\geq 0} I_{\lambda,\mu}(tu_{\lambda,\mu}, tv_{\lambda,\mu}) < \frac{N+4-\alpha}{2(2N-\alpha)} \left(\frac{\|H^+\|^{-2}_\infty}{2}\right)^{\frac{N-4}{N+4-\alpha}} S_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}} - K_0 \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right) := c_\infty.$$

Furthermore, $k^-_{\lambda,\mu} < c_\infty$ for all $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon$.

Proof. For this, we first define the functional $\mathcal{E} : \mathcal{H} \to \mathbb{R}$ such that

$$\mathcal{E}(u,v) = \frac{1}{2}\|(u,v)\|^2 - \frac{1}{2^*_\alpha} Q(u,v),\ \forall\ (u,v) \in \mathcal{H}.$$

We define the function $\phi(t) = \mathcal{E}(tU_0, tU_0)$. Take $U_0 = V_0 = \overline{U}_\epsilon$ with $(U_0, V_0) \in \mathcal{H}$. Then $\phi(t)$ satisfies $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$ small and $\phi(t) < 0$ for $t > 0$ large.

Further, one can easily verify that $\phi(t)$ attains its maximum at

$$t = \left(\frac{\|(U_0, V_0)\|^2}{2Q(U_0, V_0)} \right)^{\frac{1}{22^*_\alpha - 2}} := t^*.$$

Thus from (3.1.4), we have

$$\begin{aligned}
&\sup_{t \geq 0} \mathcal{E}(tU_0, tV_0) \\
&= \frac{(t^*)^2}{2} \|(U_0, V_0)\|^2 - \frac{(t^*)^{22^*_\alpha}}{2^*_\alpha} Q(U_0, V_0) = \frac{N + 4 - \alpha}{2N - \alpha} \left[\frac{\|\overline{U}_\epsilon\|^2}{(Q(\overline{U}_\epsilon, \overline{U}_\epsilon))^{\frac{1}{2^*_\alpha}}} \right]^{\frac{2N-\alpha}{N+4-\alpha}} \\
&\leq \frac{(N + 4 - \alpha)}{2N - \alpha} \left[\frac{(C(N, \alpha))^{\frac{N(N-4)}{4(2N-\alpha}} S_{H,L}^{\frac{N}{4}} + o(\epsilon^{N-4})}{\|H^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}} (C(N, \alpha))^{\frac{N(N-4)}{4(2N-\alpha}} S_{H,L}^{\frac{N-4}{4}} - o(\epsilon^{2N-\alpha}) - o\left(\epsilon^{\frac{2N-\alpha}{2}}\right)} \right]^{\frac{2N-\alpha}{N+4-\alpha}} \\
&\leq \frac{(N + 4 - \alpha)}{2N - \alpha} \left[\frac{\|H^+\|_\infty^{-\frac{2(N-4)}{2N-\alpha}} S_{H,L} + (\epsilon^{N-4})}{1 - o\left(\epsilon^{\frac{2N-\alpha}{2}}\right)} \right]^{\frac{2N-\alpha}{N+4-\alpha}} \\
&\leq \frac{(N + 4 - \alpha)}{2N - \alpha} \|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} S_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}} \left[1 + o(\epsilon^{N-4}) + o\left(\epsilon^{\frac{2N-\alpha}{2}}\right) \right]^{\frac{2N-\alpha}{N+4-\alpha}} \\
&\leq \frac{(N + 4 - \alpha)}{2N - \alpha} \|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} \left(\frac{\overline{S}_{H,L}}{2} \right)^{\frac{2N-\alpha}{N+4-\alpha}} + \begin{cases} o(\epsilon^{N-4}), & \alpha \leq 8 \\ o(\epsilon^{\frac{2N-\alpha}{2}}), & \alpha > 8. \end{cases} \qquad (3.5.38)
\end{aligned}$$

Further, $\delta_1 > 0$ is chosen in such a way that $c_\infty > 0$ for all $0 < (\lambda \|F\|_\beta)^{\frac{2}{2-r}} + (\mu \|G\|_\beta)^{\frac{2}{2-r}} < \delta_1$. Then, the definition of $I_{\lambda,\mu}$ and $\lambda, \mu > 0$ yield that $I_{\lambda,\mu}(tU_0, tV_0) \leq \frac{t^2}{2} \|(U_0, V_0)\|^2$ for $t \geq 0$. This implies that, there exists $t_0 \in (0, 1)$ such that

$$\sup_{t \in [0, t_0]} I_{\lambda,\mu}(tU_0, tV_0) < c_\infty \ \forall \ 0 < (\lambda \|F\|_\beta)^{\frac{2}{2-r}} + (\mu \|G\|_\beta)^{\frac{2}{2-r}} < \delta_1.$$

Moreover,

$$\sup_{t \geq t_0} P_{\lambda,\mu}(tU_0, tV_0) = \sup_{t \geq t_0} \left(\int_\Omega \lambda F(x) |tU_0|^r + \mu G(x) |tV_0|^r \right)$$

$$
\begin{aligned}
&= \sup_{t \geq t_0} \left(t^r \int_\Omega (\lambda F(x) + \mu G(x)) \, |\overline{U}_\epsilon|^r dx \right) \\
&\geq (t_0)^r \, (\lambda a_0 + \mu b_0) \int_{B(0,2r_0)} |\overline{U}_\epsilon|^r dx \\
&\geq \frac{\omega}{r} \, (\lambda + \mu) \begin{cases} o(\epsilon^{N - \frac{N-4}{2} r} |\ln \epsilon|), & r = \frac{N}{N-4} \\ o(\epsilon^{N - \frac{N-4}{2} r}), & r > \frac{N}{N-4}, \end{cases}
\end{aligned} \tag{3.5.39}
$$

where $\omega = \min\{a_0, b_0\}$.

Thus, on using (2.6.45), (3.5.38) and (3.5.39), we have

$$
\begin{aligned}
\sup_{t \geq t_0} I_{\lambda,\mu}(tU_0, tV_0) &= \sup_{t \geq t_0} \left(\mathcal{E}(tU_0, tV_0) - \frac{1}{r} P_{\lambda,\mu}(tU_0, tV_0) \right) \\
&\leq \frac{N+4-\alpha}{2N-\alpha} \|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} \left(\frac{\overline{S}_{H,L}}{2} \right)^{\frac{2N-\alpha}{N+4-\alpha}} + \begin{cases} o(\epsilon^{N-4}), & \alpha \leq 8 \\ o(\epsilon^{\frac{2N-\alpha}{2}}), & \alpha > 8 \end{cases} \\
&\quad - \frac{\omega}{r} \, (\lambda + \mu) \begin{cases} o(\epsilon^{N - \frac{N-4}{2} r} |\ln \epsilon|), & r = \frac{N}{N-4} \\ o(\epsilon^{N - \frac{N-4}{2} r}), & r > \frac{N}{N-4}, \end{cases}
\end{aligned}
$$

or

$$
\begin{aligned}
\sup_{t \geq t_0} I_{\lambda,\mu}(tU_0, tV_0) &\leq \frac{N+4-\alpha}{2N-\alpha} \|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} \left(\frac{\overline{S}_{H,L}}{2} \right)^{\frac{2N-\alpha}{N+4-\alpha}} + o(\epsilon^\rho) \\
&\quad - \frac{\omega}{r} \, (\lambda + \mu) \begin{cases} o(\epsilon^{N - \frac{N-4}{2} r} |\ln \epsilon|), & r = \frac{N}{N-4} \\ o(\epsilon^{N - \frac{N-4}{2} r}), & r > \frac{N}{N-4}, \end{cases}
\end{aligned}
$$

where $\rho = \min\{N-4, \frac{2N-\alpha}{2}\}$. Choose $\delta_2 > 0$ in this way that $0 \leq \epsilon < \delta_2$ and take $\epsilon = \left[(\lambda \|F\|_\beta)^{\frac{2}{2-r}} + (\mu \|G\|_\beta)^{\frac{2}{2-r}} \right]^{\frac{1}{\rho}}$. Thus, we have

$$
\begin{aligned}
\sup_{t \geq t_0} I_{\lambda,\mu}(tU_0, tV_0) &\leq \frac{N+4-\alpha}{2N-\alpha} \|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} \left(\frac{\overline{S}_{H,L}}{2} \right)^{\frac{2N-\alpha}{N+4-\alpha}} + o(\mathcal{D}(\lambda, \mu)) \\
&\quad - \frac{\omega}{r} \, (\lambda + \mu) \begin{cases} o((\mathcal{D}(\lambda,\mu))^{\frac{N}{2\rho}} \, |\ln \mathcal{D}(\lambda,\mu)|), & r = \frac{N}{N-4} \\ o((\mathcal{D}(\lambda,\mu))^{\frac{N}{\rho} - \frac{N-4}{2\rho} r}), & r > \frac{N}{N-4}, \end{cases}
\end{aligned}
$$

where $\mathcal{D}(\lambda,\mu)=(\lambda\|F\|_{\beta})^{\frac{2}{2-r}}+(\mu\|G\|_{\beta})^{\frac{2}{2-r}}$.

Case(i): When $\alpha\le 8$, then $\rho=N-4$.

For $r=\frac{N}{N-4}$, we can choose $\delta_3>0$ with $0<\mathcal{D}(\lambda,\mu)<\delta_3$ such that

$$o(\mathcal{D}(\lambda,\mu))-\frac{\omega(\lambda+\mu)}{r}o\left((\mathcal{D}(\lambda,\mu))^{\frac{N}{2(N-4)}}\left|\ln\left(\mathcal{D}(\lambda,\mu)\right)\right|\right)<-K_0\left(\mathcal{D}(\lambda,\mu)\right),$$

as $\lambda,\mu\to 0$ and $|\ln(\mathcal{D}(\lambda,\mu))|\to+\infty$.

For $r>\frac{N}{N-4}$, we choose $\delta_4>0$ with $0<\mathcal{D}(\lambda,\mu)<\delta_4$ such that

$$o(\mathcal{D}(\lambda,\mu))-\frac{\omega(\lambda+\mu)}{r}o\left((\mathcal{D}(\lambda,\mu))^{\frac{N}{N-4}-\frac{r}{2}}\right)<-K_0\left(\mathcal{D}(\lambda,\mu)\right),$$

as $1+\frac{2}{2-r}\left(\frac{N}{N-4}-\frac{r}{2}\right)<\frac{2}{2-r}$ for $r>\frac{N}{N-4}$. Now, we fix $\Upsilon_*=\min\{\delta_1^{\frac{2-r}{2}},\delta_2^{\frac{(2-r)(N-4)}{2}},\delta_3^{\frac{2-r}{2}},$ $\delta_4^{\frac{2-r}{2}}\}>0$ such that

$$\sup_{t\ge t_0}I_{\lambda,\mu}(tU_0,tV_0)\le\frac{N+4-\alpha}{2N-\alpha}\|H^+\|_{\infty}^{-\frac{2(N-4)}{N+4-\alpha}}\left(\frac{\overline{S}_{H,L}}{2}\right)^{\frac{2N-\alpha}{N+4-\alpha}}-K_0(\mathcal{D}(\lambda,\mu)),$$
$$\text{for } 0<\mathcal{D}(\lambda,\mu)<\Upsilon_*.$$

Case(ii): When $\alpha>8$, then $\rho=\frac{2N-\alpha}{2}$.

For $r=\frac{N}{N-4}$, we choose $\delta_5>0$ with $0<\mathcal{D}(\lambda,\mu)<\delta_5$ such that

$$o(\mathcal{D}(\lambda,\mu))-\frac{\omega(\lambda+\mu)}{r}o\left((\mathcal{D}(\lambda,\mu))^{\frac{N}{2N-\alpha}}\left|\ln\left(\mathcal{D}(\lambda,\mu)\right)\right|\right)<-K_0\left(\mathcal{D}(\lambda,\mu)\right),$$

as $\lambda,\mu\to 0$ and $|\ln(\mathcal{D}(\lambda,\mu))|\to+\infty$.

For $r>\frac{N}{N-4}$, we choose $\delta_6>0$ with $0<\mathcal{D}(\lambda,\mu)<\delta_6$ such that

$$o(\mathcal{D}(\lambda,\mu))-\frac{\omega(\lambda+\mu)}{r}o\left((\mathcal{D}(\lambda,\mu))^{\frac{2N}{2N-\alpha}-\frac{N-4}{2N-\alpha}r}\right)<-K_0\left(\mathcal{D}(\lambda,\mu)\right),$$

as $1+\frac{2}{2-r}\left(\frac{2N}{2N-\alpha}-\frac{N-4}{2N-\alpha}r\right)<\frac{2}{2-r}$ for $r>\frac{N}{N-4}$. Fix $\Upsilon_{**}=\min\{\delta_1^{\frac{2-r}{2}},\delta_2^{\frac{(2-r)(2N-\alpha)}{2}},\delta_5^{\frac{2-r}{2}},$ $\delta_6^{\frac{2-r}{2}}\}>0$ to obtain

$$\sup_{t\geq t_0} I_{\lambda,\mu}(tU_0,tV_0) \leq \frac{N+4-\alpha}{2N-\alpha}\|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}}\left(\frac{\overline{S}_{H,L}}{2}\right)^{\frac{2N-\alpha}{N+4-\alpha}} - K_0(\mathcal{D}(\lambda,\mu)),$$
$$\text{for } 0 < \mathcal{D}(\lambda,\mu) < \Upsilon_{**}. \qquad (3.5.40)$$

Thereafter, we fix $\Upsilon = \min\{\Upsilon_*, \Upsilon_{**}\}$. Thus, we have

$$\sup_{t\geq t_0} I_{\lambda,\mu}(tU_0,tV_0) \leq \frac{N+4-\alpha}{2N-\alpha}\|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}}\left(\frac{\overline{S}_{H,L}}{2}\right)^{\frac{2N-\alpha}{N+4-\alpha}} - K_0(\mathcal{D}(\lambda,\mu)) := c_\infty,$$
$$\text{for } 0 < \mathcal{D}(\lambda,\mu) < \Upsilon.$$

Later, we show that $k^-_{\lambda,\mu} < c_\infty$ for all $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon$. By using $(Z2)$, $(Z4)$ and the definition of (U_0,V_0), we get

$$P_{\lambda,\mu}(U_0,V_0) > 0 \text{ and } Q(U_0,V_0) > 0.$$

Further, by Lemma 3.3.4, definition of $k^-_{\lambda,\mu}$ and (3.5.40), there exists $t_2(U_0,V_0) \in \mathcal{N}^-_{\lambda,\mu}$ satisfying

$$k^-_{\lambda,\mu} \leq I_{\lambda,\mu}(t_2U_0,t_2V_0)) \leq I_{\lambda,\mu}(tU_0,tV_0) \leq \frac{N+4-\alpha}{2N-\alpha}\|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}}\left(\frac{\overline{S}_{H,L}}{2}\right)^{\frac{2N-\alpha}{N+4-\alpha}}$$
$$- K_0(\mathcal{D}(\lambda,\mu)) := c_\infty,$$

for each $0 < \mathcal{D}(\lambda,\mu) < \Upsilon$.

Take $(U_0,V_0) = (u_{\lambda,\mu}, v_{\lambda,\mu})$ and with this we complete the proof. □

Lemma 3.5.2. *Assume that $(Z1)-(Z4)$ hold. Then $I_{\lambda,\mu}$ satisfies the $(PS)_{k^-_{\lambda,\mu}}$ condition for all $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \left(\frac{r}{2}\right)^{\frac{2}{2-r}}\Upsilon_1$ and has a minimizer $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ in $\mathcal{N}^-_{\lambda,\mu}$ and satisfies the following conditions:*

(i) $I_{\lambda,\mu}(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) = k^-_{\lambda,\mu} > 0$.

(ii) $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ *is a nontrivial solution of the system* $(\mathcal{D}_{\lambda,\mu})$.

Proof. By virtue of Lemma 3.4.3 (ii), for $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \left(\frac{r}{2}\right)^{\frac{2}{2-r}}\Upsilon_1$, there exists a $(PS)_{k^-_{\lambda,\mu}}$-sequence $\{(u_n, v_n)\} \subset \mathcal{N}^-_{\lambda,\mu}$ in $\mathcal{H}$ for $I_{\lambda,\mu}$. Then, from Lemma 3.2.2, we find that $\{(u_n, v_n)\}$ is bounded in $\mathcal{H}$. Now, using Lemma 3.2.3 and Lemma 3.5.1, $I_{\lambda,\mu}$ satisfies the $(PS)_{k^-_{\lambda,\mu}}$-condition. Then, there exists $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) \in \mathcal{H}$ such that up to subsequence $(u_n, v_n) \to (u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ in $\mathcal{H}$. Moreover, $I_{\lambda,\mu}(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) = k^-_{\lambda,\mu} > 0$ and $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$. Using the argument as applied in Lemma 3.4.4, one can easily obtain that $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ is a nontrivial solution of system $(\mathcal{D}_{\lambda,\mu})$ for $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \left(\frac{r}{2}\right)^{\frac{2}{2-r}} \Upsilon_1$. □

Proof of Theorem 3.0.2 By Lemma 3.4.4 and Lemma 3.5.2, system $(\mathcal{D}_{\lambda,\mu})$ has one solution $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$ and another solution $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$. □

Afterwards, we show that the solutions $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ and $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ are not semi-trivial. Using Lemma 3.4.4 (i) and Lemma 3.5.2 (i) respectively, we get

$$I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) < 0 \text{ and } I_{\lambda,\mu}(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) > 0. \tag{3.5.41}$$

We observe that, if $(u, 0)$ (or $(0, v)$) is a semi-trivial solution of system $(\mathcal{D}_{\lambda,\mu})$, then we have

$$\begin{cases} \Delta^2 u = \lambda F(x)|u|^{r-2}u & \text{in } \Omega, \\ u = \nabla u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5.42}$$

Now, the energy functional $I_{\lambda,\mu}(u, 0)$ corresponding to (3.5.42) is

$$I_{\lambda,\mu}(u, 0) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{r}\int_\Omega F(x)|u|^r dx = -\frac{2-r}{2r}\|u\|^2 < 0. \tag{3.5.43}$$

Thus (3.5.41) and (3.5.43), we conclude that $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ is not a semi-trivial solution. Next, we prove that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is also not a semi-trivial solution. Without loss of generality, we assume that $v^1_{\lambda,\mu} \equiv 0$. Then $u^1_{\lambda,\mu}$ is a non-trivial solution of (3.5.42) and

$$\|(u^1_{\lambda,\mu}, 0)\|^2 = \|u^1_{\lambda,\mu}\|^2 = \lambda \int_\Omega F(x)|u^1_{\lambda,\mu}|^r dx \geq 0.$$

Moreover, we choose $w \in H_0^2(\Omega) \setminus \{0\}$ such that

$$\|(0,w)\|^2 = \|w\|^2 = \mu \int_\Omega G(x)|w|^r dx > 0.$$

From Lemma 3.3.4, there exists a unique $0 < t_1 < t_{\max}(u^1_{\lambda,\mu}, w)$ such that $(t_1 u^1_{\lambda,\mu}, t_1 w) \in \mathcal{N}^+_{\lambda,\mu}$, where

$$t_{\max}(u^1_{\lambda,\mu}, w) = \left(\frac{(22^*_\alpha - r)\int_\Omega (\lambda F(x)|u^1_{\lambda,\mu}|^r + \mu G(x)|w|^r)}{(22^*_\alpha - 2)\|(u^1_{\lambda,\mu}, w)\|^2} \right)^{\frac{1}{2-r}} = \left(\frac{22^*_\alpha - r}{22^*_\alpha - 2} \right)^{\frac{1}{2-r}} > 1.$$

Furthermore,

$$I_{\lambda,\mu}(t_1 u^1_{\lambda,\mu}, t_1 w) = \inf_{0 \le t \le t_{\max}} I_{\lambda,\mu}(t u^1_{\lambda,\mu}, tw).$$

This together with the fact that $(u^1_{\lambda,\mu}, 0) \in \mathcal{N}^+_{\lambda,\mu}$ imply that

$$\mu^+_{\lambda,\mu} \le I_{\lambda,\mu}(t_1 u^1_{\lambda,\mu}, t_1 w) \le I_{\lambda,\mu}(u^1_{\lambda,\mu}, w) < I_{\lambda,\mu}(u^1_{\lambda,\mu}, 0) = \mu^+_{\lambda,\mu},$$

which is a contradiction. Hence, $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is not a semi-trivial solution.

3.6 Conclusion

In this chapter, we establish the existence of at least two nontrivial solutions for biharmonic systems involving critical Hartree-type nonlinearity with sign-changing weight functions with respect to the pair of parameters λ, μ belongs to a suitable subset of $\mathbb{R}^2$. The main aspect of this chapter is the study of the critical level (c_∞) below which the Palais-Smale condition is satisfied. Altogether, this work contributes to the study of elliptic systems with critical Choquard nonlinearity having sign-changing weight functions. The results obtained here are new for the Laplacian case also. Moreover, one can generalize the work of chapters 2 and 3 for polyharmonic operators.

4

p-Biharmonic Equation Involving Choquard Nonlinearity

In this chapter, we establish the existence and multiplicity results of the nontrivial solutions for p-biharmonic equations with subcritical or critical Choquard nonlinearity involving sign-changing weight functions. To discuss the critical case, one needs to study the concentration-compactness Lemma for Choquard type nonlinearity in case of p-bihamonic operator, which is never studied earlier even in p-Laplacian case. With this motivation, we consider the following p-biharmonic Choquard equation with subcritical or critical nonlinearities

$$(\mathcal{G}_\lambda)\begin{cases} \Delta_p^2 u = \lambda f(x)|u|^{r-2}u + g(x)\left(\displaystyle\int_\Omega \frac{g(y)|u(y)|^q}{|x-y|^\alpha}dy\right)|u|^{q-2}u & \text{in } \Omega, \\ u, \nabla u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain having smooth boundary $\partial\Omega$, $p \in (1, \infty)$, $N > 2p$, $1 < r < p$, $0 < \alpha < N$, $\frac{p(2N-\alpha)}{2N} \le q \le \frac{p(2N-\alpha)}{2(N-2p)}$, $p_\alpha^* := \frac{p(2N-\alpha)}{2(N-2p)}$ is the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality, $\Delta_p^2 u := \Delta\left(|\Delta u|^{p-2}\Delta u\right)$ denotes the p-biharmonic operator and $\lambda > 0$ is a parameter.
Before stating our main results, we require the following assumptions on sign-changing weight functions f and g respectively:

$(f1)$ $f \in L^\beta(\Omega)$, where $\beta = \frac{p^*}{p^*-r}$ with $p^* = \frac{Np}{N-2p}$ and $f^+ = \max\{f, 0\} \not\equiv 0$ in Ω.
$(g1)$ $g \in C(\overline{\Omega})$ and $g^+ = \max\{g, 0\} \not\equiv 0$ in Ω.

Set

$$\Upsilon_1 = \left[\left(\frac{p-r}{2q-r}\right)\|g^+\|_\infty^{-2}(C(N,\alpha))^{-1}S^{\frac{2q}{p}}\right]^{\frac{p-r}{2q-p}}\left(\frac{2q-p}{2q-r}\right)\|f\|_\beta^{-1}S^{\frac{r}{p}}.$$

Theorem 4.0.1. *(Existence in subcritical case:) Suppose the assumptions $(f1)$ and $(g1)$ hold. If $1 < r < p$ and $\frac{p(2N-\alpha)}{2N} \le q < p_\alpha^*$, then there exists $\Upsilon_1 > 0$ such that the problem $(\mathcal{G}_\lambda)$ has at least one nontrivial solution in $\mathcal{N}_\lambda^+$ for every $\lambda \in (0, \Upsilon_1)$.*

Theorem 4.0.2. *(Multiplicity in subcritical case:) Suppose the assumptions $(f1)$ and $(g1)$ hold. If $1 < r < p$ and $\frac{p(2N-\alpha)}{2N} \le q < p_\alpha^*$, then there exists $\Upsilon_2 > 0$ such that the problem $(\mathcal{G}_\lambda)$ has at least one nontrivial solution in $\mathcal{N}_\lambda^-$ for every $\lambda \in (0, \Upsilon_2)$.*

Theorem 4.0.3. *(Existence in the critical case:) Suppose that assumptions $(f1)$ and $(g1)$ hold. If $1 < r < p$ and $\frac{p(2N-\alpha)}{2N} \le q = p_\alpha^*$, then there exists $\Upsilon_1 > 0$ such that the problem $(\mathcal{G}_\lambda)$ has at least one nontrivial solution in $\mathcal{N}_\lambda^+$ for every $\lambda \in (0, \Upsilon_1)$.*

4.1 Preliminaries

In this section, we give the preliminary results and prove some required Lemmas which are applied to obtain the main results.
Suppose $g = h = |u|^s$, then $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^s|u(y)|^s}{|x-y|^\alpha}dxdy$ is well defined by Hardy-Littlewood-Sobolev inequality for $|u|^s \in L^t(\mathbb{R}^N)$ with $t > 1$ and $\frac{2}{t} + \frac{\alpha}{N} = 2$. Thus,

for $u \in W^{2,p}(\mathbb{R}^N)$, s satisfies the following

$$\frac{p(2N-\alpha)}{2N} \leq s \leq \frac{p(2N-\alpha)}{2(N-2p)} := p_\alpha^*,$$

where $\frac{p(2N-\alpha)}{2N}$ and p_α^* are known as lower and upper critical exponent respectively in the sense of Hardy-Littlewood-Sobolev inequality.
Consider $U(x) = \nu \left(1 + |x|^{\frac{p}{p-1}}\right)^{-\frac{N-2p}{p}}$, then $U_\epsilon(x) = \epsilon^{\frac{2p-N}{p}} U\left(\frac{x}{\epsilon}\right)$, where $\epsilon > 0$ provides the family of minimizers for S, where S is the best Sobolev constant in the embedding of $W_0^{2,p}(\Omega)$ into $L^{p^*}(\Omega)$ with $p^* = \frac{Np}{N-2p}$, defined by

$$S := \inf_{u \in X \setminus \{0\}} \frac{\int_\Omega |\Delta u|^p dx}{(\int_\Omega |u(x)|^{p^*} dx)^{\frac{p}{p^*}}}.$$

We define $S_{H,L}$ to be the best constant as

$$S_{H,L} := \inf_{u \in D^{2,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\Delta u|^p dx}{\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{p_\alpha^*} |u(y)|^{p_\alpha^*}}{|x-y|^\alpha} dxdy\right)^{\frac{p}{2p_\alpha^*}}}.$$

Now, for all $u \in D^{2,p}(\mathbb{R}^N)$, by Hardy-Littlewood-Sobolev inequality, it is observed that

$$\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{p_\alpha^*} |u(y)|^{p_\alpha^*}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{p_\alpha^*}} \leq (C(N,\alpha))^{\frac{1}{p_\alpha^*}} \|u\|_{p^*}^2, \tag{4.1.1}$$

where $C(N,\alpha)$ is same as defined in Proposition 1.1.1. From above, we conclude the following relation between $S_{H,L}$ and S.

$$S_{H,L} \geq (C(N,\alpha))^{-\frac{p}{2p_\alpha^*}} S. \tag{4.1.2}$$

Lemma 4.1.1. *Let $N > 2p$, $0 < \alpha < N$ and $\{u_n\}$ be a bounded sequence in $L^{p^*}(\Omega)$, then*

$$\left(\int_\Omega \frac{|u_n(y)|^{p_\alpha^*} |u_n(x)|^{p_\alpha^*-2}}{|x-y|^\alpha} dy\right) u_n(x) \rightharpoonup \left(\int_\Omega \frac{|u(y)|^{p_\alpha^*} |u(x)|^{p_\alpha^*-2}}{|x-y|^\alpha} dy\right) u(x)$$

weakly in $L^{\frac{Np}{Np-N+2p}}(\Omega)$ *as* $n \to \infty$.

Proof. The proof is alike to the proof of Lemma 2.2 [32]. But we demonstrate it here for the completeness. Let $\{u_n\}$ be a bounded sequence in $L^{p^*}(\Omega)$, then one can easily verify that

$$\begin{aligned} |u_n|^{p^*_\alpha} &\rightharpoonup |u|^{p^*_\alpha} \text{ weakly in } L^{\frac{2N}{2N-\alpha}}(\Omega), \\ |u_n|^{p^*_\alpha-2}u_n &\rightharpoonup |u|^{p^*_\alpha-2}u \text{ weakly in } L^{\frac{2Np}{2Np-\alpha p-2N+4p}}(\Omega), \end{aligned} \tag{4.1.3}$$

as $n \to \infty$. The Riesz potential defines a continuous map from $L^{\frac{2N}{2N-\alpha}}(\Omega)$ to $L^{\frac{2N}{\alpha}}(\Omega)$, by Hardy-Littlewood-Sobolev inequality. Thus, we have

$$\int_\Omega \frac{|u_n(y)|^{p^*_\alpha}}{|x-y|^\alpha}dy \rightharpoonup \int_\Omega \frac{|u(y)|^{p^*_\alpha}}{|x-y|^\alpha}dy \text{ weakly in } L^{\frac{2N}{\alpha}}(\Omega), \tag{4.1.4}$$

as $n \to \infty$. Then, on combining (4.1.3) and (4.1.4), we obtain

$$\left(\int_\Omega \frac{|u_n(y)|^{p^*_\alpha}|u_n(x)|^{p^*_\alpha-2}}{|x-y|^\alpha}dy\right)u_n(x) \rightharpoonup \left(\int_\Omega \frac{|u(y)|^{p^*_\alpha}|u(x)|^{p^*_\alpha-2}}{|x-y|^\alpha}dy\right)u(x) \text{ in } L^{\frac{Np}{Np-N+2p}}(\Omega)$$

as $n \to \infty$, which is the required result. □

4.2 Concentration-compactness principle

The main obstacle in solving the problem $(\mathcal{G}_\lambda)$ is due to lack of compactness in the embedding of $W_0^{2,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$. To tackle such type of problem, P. L. Lions established the concentration-compactness principles [50, 51]. In this section, we will prove the application of concentration-compactness principle in sense of critical Choquard equation. The idea of proof is taken from [50]. For reader's convenience, will give the detail of proof. First of all, we would like to recall the second concentration-compactness principle [50] which is as follows.

Lemma 4.2.1. *Let* $\{u_n\}$ *be a bounded sequence in* $D^{2,p}(\mathbb{R}^N)$ *converging weakly and a.e. to* $u \in D^{2,p}(\mathbb{R}^N)$ *such that* $|\Delta u_n|^p \rightharpoonup \mu$, $|u_n|^{p^*} \rightharpoonup \omega$ *in the sense of measure.*

Then, for at most countable set J, there exist two family of distinct points $\{\omega_i : i \in J\}$ and $\{\mu_i : i \in J\}$ in $\mathbb{R}^N$ satisfying

$$\omega = |u|^{p^*} + \sum_{i \in J} \omega_i \delta_{x_i}, \ \omega_i > 0,$$
$$\mu \geq |\Delta u|^p + \sum_{i \in J} \mu_i \delta_{x_i}, \ \mu_i > 0,$$
$$S\omega_i^{\frac{p}{p^*}} \leq \mu_i, \ \forall i \in J,$$

where μ, ω are bounded and nonnegative measures on $\mathbb{R}^N$ and δ_{x_i} is the Dirac mass at x_i. In particual, $\sum_{i \in J} (\omega_i)^{\frac{p}{p^*}} < \infty$.

In order to prove the existence of nontrivial solution, we will show that the lack of compactness is encountered by applying concentration-compactness principle. The concentration-compactness principle assists us in two manner. Firstly, it examines the nature of weakly convergent sequences in Sobolev spaces in which the lack of compactness arises because of critical Sobolev exponent. Secondly, it provides the level below which the corresponding energy functional satisfies the Palais-Smale condition. Therefore, we set down the proof in frame using the same concept given in [50, 51].

Lemma 4.2.2. *Assume that $\{u_n\}$ be a bounded sequence in $D^{2,p}(\mathbb{R}^N)$ converging weakly and a.e. to $u \in D^{2,p}(\mathbb{R}^N)$. Suppose*

$$\left(\int_\Omega \frac{|u_n(y)|^{p^*_\alpha}}{|x-y|^\alpha} dy \right) |u_n(x)|^{p^*_\alpha} \rightharpoonup \eta,$$

in the sense of measure. Then there exists a countable sequence of points $\{x_i\}_{i \in J} \subset \mathbb{R}^N$ and families of positive numbers $\{\mu_i : i \in J\}$, $\{\omega_i : i \in J\}$ and $\{\eta_i : i \in J\}$ such that

$$\eta = \left(\int_\Omega \frac{|u(y)|^{p_\alpha^*}}{|x-y|^\alpha}dy\right)|u(x)|^{p_\alpha^*} + \sum_{i\in J}\eta_i\delta_{x_i}, \tag{4.2.5}$$

$$\mu \geq |\Delta u|^p + \sum_{i\in J}\mu_i\delta_{x_i}, \tag{4.2.6}$$

$$\omega \geq |u|^{p^*} + \sum_{i\in J}\omega_i\delta_{x_i}, \tag{4.2.7}$$

$$S_{H,L}\eta_i^{\frac{N-2p}{2N-\alpha}} \leq \mu_i,\ \eta_i \leq C(N,\alpha)\omega_i^{\frac{2N-\alpha}{N}}, \tag{4.2.8}$$

where η, μ and ω are bounded and nonnegative measures on $\mathbb{R}^N$ and δ_{x_i} is the Dirac mass concentrated at $x_i \in \mathbb{R}^N$.

Proof. Let $v_n = u_n - u$. Then $\{v_n\}$ converging weakly to 0 in $D^{2,p}(\mathbb{R}^N)$ and $v_n(x) \to 0$ a.e. in $\mathbb{R}^N$ as the bounded sequence $\{u_n\}$ converging weakly to u in $D^{2,p}(\mathbb{R}^N)$. By Lemma 4.2.1, we can write

$$\left(\int_{\mathbb{R}^N} \frac{|v_n(y)|^{p_\alpha^*}}{|x-y|^\alpha}dy\right)|v_n(x)|^{p_\alpha^*} \rightharpoonup \tau_1 := \eta - \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{p_\alpha^*}}{|x-y|^\alpha}dy\right)|u(x)|^{p_\alpha^*},$$
$$|\Delta v_n|^p \rightharpoonup \tau_2 := \mu - |\Delta u|^p,$$
$$|v_n|^{p^*} \rightharpoonup \tau_3 := \omega - |u|^{p^*}.$$

Firstly, we will show that for every $\phi \in C_c^\infty(\mathbb{R}^N)$,

$$\left|\int_{\mathbb{R}^N}\left(|x|^{-\alpha} * |\phi v_n(x)|^{p_\alpha^*}\right)|\phi v_n(x)|^{p_\alpha^*} - \int_{\mathbb{R}^N}\left(|x|^{-\alpha} * |v_n(x)|^{p_\alpha^*}\right)|\phi(x)|^{p_\alpha^*}|\phi v_n(x)|^{p_\alpha^*}\right| \to 0. \tag{4.2.9}$$

For this,
denote $\psi_n(x) := \left[\left(|x|^{-\alpha} * |\phi v_n(x)|^{p_\alpha^*}\right) - \left(|x|^{-\alpha} * |v_n(x)|^{p_\alpha^*}\right)|\phi(x)|^{p_\alpha^*}\right]|\phi v_n(x)|^{p_\alpha^*}$.
Since $\phi \in C_c^\infty(\mathbb{R}^N)$, we have for every $\delta > 0$, there exists $K > 0$ such that

$$\int_{|x|\geq K} |\psi_n(x)|dx < \delta \ \ \forall\, n \geq 1. \tag{4.2.10}$$

Further, we know that Riesz potential defines a linear operator and by using $v_n(x) \to 0$ a.e. in $\mathbb{R}^N$, we obtain

$$\int_{\mathbb{R}^N} \frac{|v_n(y)|^{p^*_\alpha}}{|x-y|^\alpha} dy \to 0 \text{ a.e. in } \mathbb{R}^N.$$

Thus $\psi_n(x) \to 0$ a.e. in $\mathbb{R}^N$. We note that

$$\begin{aligned} \psi_n(x) &= \int_{\mathbb{R}^N} \frac{\left(|\phi(y)|^{p^*_\alpha} - |\phi(x)|^{p^*_\alpha}\right)}{|x-y|^\alpha} |v_n(y)|^{p^*_\alpha} dy |\phi v_n(x)|^{p^*_\alpha} \\ &:= \int_{\mathbb{R}^N} \Phi(x,y) |v_n(y)|^{p^*_\alpha} dy |\phi v_n(x)|^{p^*_\alpha}, \end{aligned}$$

where $\Phi(x,y) = \frac{|\phi(y)|^{p^*_\alpha} - |\phi(x)|^{p^*_\alpha}}{|x-y|^\alpha}$. Moreover, for almost all $W^{2,p}_0(\Omega)$, there exists some $R > 0$ large enough such that

$$\int_{\mathbb{R}^N} \Phi(x,y)|v_n(y)|^{p^*_\alpha} dy = \int_{|y|\leq R} \Phi(x,y)|v_n(y)|^{p^*_\alpha} dy - |\phi(x)|^{p^*_\alpha} \int_{|y|\geq R} \frac{|v_n(y)|^{p^*_\alpha}}{|x-y|^\alpha} dy.$$

In [51], we noticed that $\Phi(x,y) \in L^q(B_R)$ for each $W^{2,p}_0(\Omega)$, where $q < \frac{N}{\alpha-1}$ if $\alpha > 1$, $q \leq +\infty$ if $0 < \alpha \leq 1$. So, by Young's inequality, there exists $t > \frac{2N}{\alpha}$ such that

$$\left(\int_{B_K} \left(\int_{B_R} \Phi(x,y)|v_n(y)|^{p^*_\alpha} dy \right)^t dx \right)^{\frac{1}{t}} \leq L_\phi \|\Phi(x,y)\|_q \| |v_n|^{p^*_\alpha} \|_{\frac{2N}{2N-\alpha}} \leq L'_\phi,$$

where K is same as in (4.2.10). Moreover, one can easily see that for $R > 0$ large enough

$$\left(\int_{B_K} \left(|\phi(x)|^{p^*_\alpha} \int_{|y|\geq R} \frac{|v_n(y)|^{p^*_\alpha}}{|x-y|^\alpha} dy \right)^t dx \right)^{\frac{1}{t}} \leq L,$$

and so, we have

$$\left(\int_{B_K} \left(\int_{\mathbb{R}^N} \Phi(x,y)|v_n(y)|^{p^*_\alpha} dy \right)^t dx \right)^{\frac{1}{t}} \leq L''_\phi.$$

Thus for $s > 0$ small enough, we obtain

$$\int_{B_K} |\psi_n(x)|^{1+s} dx \leq \left(\int_{B_K} \left(\int_{\mathbb{R}^N} \Phi(x,y) |v_n(y)|^{p^*_\alpha} dy \right)^t dx \right)^{\frac{1}{t}} \left(\int_{B_K} |\phi v_n|^{p^*} dx \right)^{\frac{p^*_\alpha}{p^*}} \leq L''_\phi.$$

Using this together with $\psi_n(x) \to 0$ a.e. in $\mathbb{R}^N$, we have

$$\int_{B_K} |\psi_n(x)| dx \to 0 \text{ as } n \to \infty.$$

Combining this with (4.2.10), we have

$$\int_{\mathbb{R}^N} |\psi_n(x)| dx \to 0 \text{ as } n \to \infty.$$

Now, for every $\phi \in C_c^\infty(\mathbb{R}^N)$, by Hardy-Littlewood-Sobolev inequality, we obtain

$$\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{|\phi v_n(y)|^{p^*_\alpha}}{|x-y|^\alpha} dy \right) |\phi v_n(x)|^{p^*_\alpha} dx \leq C(N,\alpha) \|\phi v_n\|_{p^*}^{2p^*_\alpha}.$$

From equation (4.2.9), we have

$$\int_{\mathbb{R}^N} |\phi(x)|^{2p^*_\alpha} \left(\int_{\mathbb{R}^N} \frac{|v_n(y)|^{p^*_\alpha}}{|x-y|^\alpha} dy \right) |v_n(x)|^{p^*_\alpha} dx \leq C(N,\alpha) \|\phi v_n\|_{p^*}^{2p^*_\alpha}.$$

On taking the limit as $n \to \infty$, we obtain

$$\int_{\mathbb{R}^N} |\phi(x)|^{2p^*_\alpha} d\tau_1 \leq C(N,\alpha) \left(\int_{\mathbb{R}^n} |\phi|^{p^*} d\tau_3 \right)^{\frac{2p^*_\alpha}{p^*}}. \tag{4.2.11}$$

Applying Lemma 4.2.1, one can directly obtain (4.2.6) and (4.2.7).
Further, let $\phi = \chi_{\{x_i\}}$, $i \in J$ and using this in (4.2.11), we have

$$\eta_i^{\frac{p^*}{2p^*_\alpha}} \leq (C(N,\alpha))^{\frac{p^*}{2p^*_\alpha}} w_i, \ \forall\, i \in J.$$

Definition of $S_{H,L}$ yields

$$\left(\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|\phi v_n(y)|^{p_\alpha^*}}{|x-y|^\alpha}dy\right)|\phi v_n(x)|^{p_\alpha^*}dx\right)^{\frac{p}{2p_\alpha^*}}S_{H,L}\leq\int_{\mathbb{R}^N}|\Delta(\phi v_n)|^p dx.$$

Also by (4.2.9) and $v_n \to 0$ in $L^p_{loc}(\mathbb{R}^N)$, we obtain

$$\left(\int_{\mathbb{R}^N}|\phi(x)|^{2p_\alpha^*}\left(\int_{\mathbb{R}^N}\frac{|v_n(y)|^{p_\alpha^*}}{|x-y|^\alpha}dy\right)|v_n(x)|^{p_\alpha^*}dx\right)^{\frac{p}{2p_\alpha^*}}S_{H,L}\leq\int_{\mathbb{R}^N}\phi^p|\Delta v_n|^p dx+o(1).$$

On passing the limit as $n \to \infty$, we have

$$\left(\int_{\mathbb{R}^N}|\phi(x)|^{2p_\alpha^*}d\tau_1\right)^{\frac{p}{2p_\alpha^*}}S_{H,L}\leq\int_{\mathbb{R}^N}\phi^p d\tau_2. \tag{4.2.12}$$

Let $\phi = \chi_{\{x_i\}}$, $i \in J$ and applying this in (4.2.12), we have

$$S_{H,L}\eta_i^{\frac{p}{2p_\alpha^*}} \leq \mu_i,\ \forall\, i \in J.$$

This completes the proof of (4.2.5) and (4.2.8). □

In the end of this section, we define the Euler functional corresponding to the problem $(\mathcal{G}_\lambda)$ as follows

$$\mathcal{I}_\lambda(u)=\frac{1}{p}\|u\|^p-\frac{1}{2q}\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^q|u(y)|^q}{|x-y|^\alpha}dxdy-\frac{\lambda}{r}\int_\Omega f(x)|u|^r dx. \tag{4.2.13}$$

For problem $(\mathcal{G}_\lambda)$, $\mathcal{I}_\lambda$ is well defined in $W_0^{2,p}(\Omega)$. Moreover, u is a weak solution of the problem if and only if u becomes the critical point of the functional $\mathcal{I}_\lambda$.

4.3 Fibering map analysis

In this section, we discuss Nehari manifold, fiber map, and prove some important results which are used to obtain the main results. We clearly observe that the energy functional $\mathcal{I}_\lambda$ is unbounded below on $W_0^{2,p}(\Omega)$. So we restrict $\mathcal{I}_\lambda$ to a appropriate subset of $W_0^{2,p}(\Omega)$, known as Nehari set and introduced as

$$\mathcal{N}_\lambda = \left\{u \in W_0^{2,p}(\Omega) \setminus \{0\} : \langle \mathcal{I}'_\lambda(u), u\rangle = 0\right\}.$$

This implies that $u \in \mathcal{N}_\lambda$ if and only if

$$\langle \mathcal{I}'_\lambda(u), u\rangle = \|u\|^p - \int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^q|u(y)|^q}{|x-y|^\alpha} - \lambda\int_\Omega f(x)|u|^r = 0. \tag{4.3.14}$$

Therefore, $\mathcal{N}_\lambda$ contains every nontrivial solution corresponding to the problem $(\mathcal{G}_\lambda)$. Now we will show that $\mathcal{I}_\lambda$ is bounded below and coercive on $\mathcal{N}_\lambda$ by the following Lemma.

Lemma 4.3.1. *The energy functional $\mathcal{I}_\lambda$ is coercive and bounded on $\mathcal{N}_\lambda$.*

Proof. Let $u \in \mathcal{N}_\lambda$. Then using (4.3.14), Hölder's inequality and Sobolev embedding, we have

$$\begin{aligned}\mathcal{I}_\lambda(u) &= \left(\frac{1}{p} - \frac{1}{2q}\right)\|u\|^p - \lambda\left(\frac{1}{r} - \frac{1}{2q}\right)\int_\Omega f(x)|u|^r dx \\ &\geq \left(\frac{1}{p} - \frac{1}{2q}\right)\|u\|^p - \lambda\left(\frac{1}{r} - \frac{1}{2q}\right)S^{-\frac{r}{p}}\|f\|_\beta\|u\|^r \end{aligned} \tag{4.3.15}$$

where $\beta = \frac{p^*}{p^*-r}$. Since $1 < r < p$, we can conclude that $\mathcal{I}_\lambda$ is coercive.
Now, define the function $\xi : \mathbb{R}^+ \to \mathbb{R}$ such that $\xi(t) = At^p - Bt^r$. Then, it is easy to calculate that ξ achieves its minimum at $t = \left(\frac{Br}{Ap}\right)^{\frac{1}{p-r}} := t^*$. Also, $\xi(t) \geq \xi(t^*) := -(p-r)\left(\frac{B}{p}\right)^{\frac{p}{p-r}}\left(\frac{r}{A}\right)^{\frac{r}{p-r}}$. Thus, taking $A = \frac{1}{p} - \frac{1}{2q}$, $B = \lambda\left(\frac{1}{r} - \frac{1}{2q}\right)S^{-\frac{r}{p}}\|f\|_\beta$ and $t = \|u\|$ in the definition of ξ, we obtain

$$\mathcal{I}_\lambda(u) \geq \xi(\|u\|) \geq \xi(t^*) := -(p-r)\frac{(2q-r)^{\frac{p}{p-r}}}{(2q-p)^{\frac{r}{p-r}}}\left(S^{-\frac{r}{p}}\|f\|_\beta\lambda\right)^{\frac{p}{p-r}}.$$

This shows that $\mathcal{I}_\lambda$ is bounded below on $\mathcal{N}_\lambda$, which completes the proof. □

Lemma 4.3.2. *If u is the local minimizer for $\mathcal{I}_\lambda$ on subsets $\mathcal{N}_\lambda^+$ or $\mathcal{N}_\lambda^-$ of $\mathcal{N}_\lambda$ such that $u \notin \mathcal{N}_\lambda^0$. Then $\mathcal{I}'_\lambda(u) = 0$ in $(W_0^{2,p}(\Omega))^{-1}$, where $(W_0^{2,p}(\Omega))^{-1}$ denotes the dual space of $W_0^{2,p}(\Omega)$.*

Proof. The proof follows the same as done in Lemma 1.3.1 in chapter 1. $\square$

The Nehari manifold is firmly related to the behavior of the map $\Psi_u : t \to \mathcal{I}_\lambda(tu)$, known as fibering maps and introduced by Drabek and Pohozaev [24]. For $u \in W_0^{2,p}(\Omega)$, we have

$$\begin{aligned}
&\Psi_u(t) = \mathcal{I}_\lambda(tu) = \frac{t^p}{p}\|u\|^p - \frac{t^{2q}}{2q}G(u) - \frac{\lambda t^r}{r}F(u),\\
&\Psi'_u(t) = t^{p-1}\|u\|^p - t^{2q-1}G(u) - \lambda t^{r-1}F(u),\\
&\Psi''_u(t) = (p-1)t^{p-2}\|u\|^p - (2q-1)t^{2q-2}G(u) - \lambda(r-1)t^{r-2}F(u),
\end{aligned}$$

where $F(u) := \int_\Omega f(x)|u|^r dx$ and $G(u) := \int_\Omega \int_\Omega g(x)g(y)\frac{|u(x)|^q|u(y)|^q}{|x-y|^\alpha}dxdy$ respectively. It is observed that $tu \in \mathcal{N}_\lambda$ if and only if $\Psi'_u(t) = 0$. Particularly $u \in \mathcal{N}_\lambda$ if and only if $\Psi'_u(1) = 0$. So it is natural to break $\mathcal{N}_\lambda$ into three subsets $\mathcal{N}_\lambda^+, \mathcal{N}_\lambda^-$ and $\mathcal{N}_\lambda^0$ corresponding to local minima, local maxima and point of inflection respectively. Thus, we have

$$\mathcal{N}_\lambda^\pm := \{u \in \mathcal{N}_\lambda : \Psi''_u(1) \gtrless 0\} \text{ and } \mathcal{N}_\lambda^0 := \{u \in \mathcal{N}_\lambda : \Psi''_u(1) = 0\}.$$

Further for $u \in \mathcal{N}_\lambda$, we have

$$\Psi''_u(1) = \begin{cases} (p-2q)\|u\|^p - \lambda(r-2q)\int_\Omega f(x)|u|^r dx > 0 \\ (p-r)\|u\|^p - (2q-r)\int_\Omega \int_\Omega g(x)g(y)\frac{|u(x)|^q|u(y)|^q}{|x-y|^\alpha}dxdy. \end{cases} \tag{4.3.16}$$

Lemma 4.3.3. *We have the following:*
(i) If $u \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0$, then $\int_\Omega f(x)|u|^r dx > 0$;
(ii) If $u \in \mathcal{N}_\lambda^- \cup \mathcal{N}_\lambda^0$, then $\int_\Omega \int_\Omega g(x)g(y)\frac{|u(x)|^q|u(y)|^q}{|x-y|^\alpha}dxdy > 0$.

Proof. Proof directly follows from (4.3.16), and the definition of $\mathcal{N}_\lambda^\pm$ and $\mathcal{N}_\lambda^0$. $\square$

Lemma 4.3.4. *Let $0 < \lambda < \Upsilon_1$, where Υ_1 is same as in (4.3.17), $u \in W_0^{2,p}(\Omega)$, $F(u) := \int_\Omega f(x)|u|^r$ and $G(u) := \int_\Omega \int_\Omega g(x)g(y)\frac{|u(x)|^q|u(y)|^q}{|x-y|^\alpha}$, then we have the following results:*

(i) If $F(u) < 0$ and $G(u) < 0$, then there does not exist any critical point.

(ii) *If $F(u) > 0$ and $G(u) \leq 0$, then there exists a unique $t^+(u)$ such that $t^+u \in \mathcal{N}_\lambda^+$ and $\mathcal{I}_\lambda(t^+u) = \inf_{t\geq 0} \mathcal{I}_\lambda(tu)$.*

(iii) *If $G(u) > 0$ and $F(u) \leq 0$, then there exists a unique $t^-(u) > t_{\max}$ such that $t^-(u)u \in \mathcal{N}_\lambda^-$ and $\mathcal{I}_\lambda(t^-u) = \sup_{t\geq t_{\max}} \mathcal{I}_\lambda(tu)$.*

(iv) *If $G(u) > 0$ and $F(u) > 0$, then there exists unique t^+ and t^- satisfying $0 < t^+ < t_{\max} < t^-$ such that $t^+(u)u \in \mathcal{N}_\lambda^+$ and $t^-(u)u \in \mathcal{N}_\lambda^-$. Moreover*

$$\mathcal{I}_\lambda(t^+u) = \inf_{0\leq t\leq t_{\max}} \mathcal{I}_\lambda(tu);\ \mathcal{I}_\lambda(t^-u) = \sup_{t\geq t_{\max}} \mathcal{I}_\lambda(tu).$$

Proof. It is noticed that the nature of fibering maps depending upon the signs of $F(u)$ and $G(u)$ respectively. So, for each $u \in W_0^{2,p}(\Omega)$, we define a map $\mathcal{K}_u : \mathbb{R}^+ \to \mathbb{R}$ by

$$\mathcal{K}_u(t) = t^{p-r}\|u\|^p - t^{2q-r}G(u).$$

One can see that, for $t > 0$, $tu \in \mathcal{N}_\lambda$ if and only if t is a solution of $\mathcal{K}_u(t) = \lambda F(u)$. Further

$$\mathcal{K}_u'(t) = (p-r)t^{p-r-1}\|u\|^p - (2q-r)t^{2q-r-1}G(u),$$

then, $\mathcal{K}_u'(t) = 0$ at $t = t_{\max}$, where

$$t_{\max} = \left(\frac{(p-r)\|u\|^p}{(2q-r)G(u)}\right)^{\frac{1}{2q-p}}.$$

Further, if $G(u) > 0$, then it is observed that $\mathcal{K}_u(0) = 0$, $\mathcal{K}_u(t) \to -\infty$ as $t \to \infty$ and $\mathcal{K}_u(t)$ attains its maximum value at $t_{\max}$. Thus

$$\begin{aligned}\mathcal{K}_u(t_{\max}) =& \|u\|^r \left(\frac{p-r}{2q-r}\right)^{\frac{p-r}{2q-p}} \left(\frac{2q-p}{2q-r}\right) \left(\frac{\|u\|^{2q}}{G(u)}\right)^{\frac{p-r}{2q-p}} \\ \geq& \left[\left(\frac{p-r}{2q-r}\right) \|g^+\|_\infty^{-2} S_{H,L}^{\frac{2q}{p}}\right]^{\frac{p-r}{2q-p}} \left(\frac{2q-p}{2q-r}\right) \|u\|^r.\end{aligned}$$

If $0 \neq u \in W_0^{2,p}(\Omega)$ satisfying

(i) $F(u) < 0$ and $G(u) < 0$, then $\Psi_u(0) = 0$, $\Psi_u'(t) > 0$ for all $t > 0$. This implies Ψ_u is strictly increasing. So no critical point exists in this case.

(ii) If $G(u) < 0$, then $\mathcal{K}_u$ is strictly increasing for $t > 0$ and $\mathcal{K}_u(0) = 0$. As $F(u) \geq 0$, so there exists a unique $t^+ = t^+(u)$ such that $\mathcal{K}_u(t^+(u)) = \lambda F(u) > 0$. Further $\Psi_u'(t) = t^r(\mathcal{K}_u(t) - \lambda F(u))$ implies that $\Psi_u'(t^+(u)) = 0$. Thus, $t^+(u)u \in \mathcal{N}_\lambda$. Also $\Psi_u'(t)$ is increasing for $t > t^+(u)$ and $\Psi_u'(t)$ is decreasing for $t < t^+(u)$ with $\Psi_u''(t^+(u)) = (t^+(u))^{r+1}\mathcal{K}_u'(t^+(u)) > 0$. This shows that Ψ_u has one critical point corresponding to local minima. Hence $t^+(u)u \in \mathcal{N}_\lambda^+$ and $\mathcal{I}_\lambda(t^+(u)u) = \inf_{t \geq 0} \mathcal{I}_\lambda(tu)$.

(iii) If $G(u) > 0$, then $\mathcal{K}_u'(t) > 0$ for $t \in [0, t_{\max})$ and $\mathcal{K}_u'(t) < 0$ for $t \in (t_{\max}, \infty)$. Thus, $\mathcal{K}_u(t)$ attains its maximum at $t_{\max}$. Since $\lambda F(u) \leq 0$, there is a unique $t^-(u) > t_{\max}(u) > 0$ such that $\mathcal{K}_u(t^-(u)) = \lambda F(u)$ and $\mathcal{K}_u(t^-) < 0$. Now, the relation $\Psi_u'(t) = t^r(\mathcal{K}_u(t) - \lambda F(u))$ implies that $\Psi_u'(t^-(u)) = 0$. Therefore, Ψ_u has one critical point corresponding to local maxima. Thus $t^-(u)u \in \mathcal{N}_\lambda$. Moreover, $\Psi_u''(t^-(u)) = (t^-(u))^{r+1}\mathcal{K}_u'(t^-(u)) < 0$ implies $t^-(u)u \in \mathcal{N}_\lambda^-$. Also $\Psi_u'(t) < 0$ for $t > t_{max}$, so $\Psi_u(t^-) = \sup_{t \geq t_{\max}} \Psi_u(t)$. Hence $\mathcal{I}_\lambda(t^- u) = \sup_{t \geq t_{\max}} \mathcal{I}_\lambda(tu)$.

(iv) Since $G(u) > 0$, $\mathcal{K}_u$ attains its maximum value at $t = t_{\max}$. Now, if $F(u) > 0$, then

$$\begin{aligned}\mathcal{K}_u(t_{\max}) - \lambda F(u) &\geq \frac{p-2q}{r-2q}\left(\frac{p-r}{2q-r}\|g^+\|_\infty^{-2} S_{H,L}^{\frac{2q}{p}}\right)^{\frac{p-r}{2q-p}} \|u\|^r - \lambda S^{-\frac{r}{p}}\|f\|_\beta \|u\|^r \\ &> 0,\end{aligned}$$

for $0 < \lambda < \Upsilon_1$, where Υ_1 is same as in (4.3.17). Thus, there exists $t^+(u)$ and $t^-(u)$ such that $0 < t^+(u) < t_{\max} < t^-(u)$,

$$\mathcal{K}_u(t^+(u)) = \lambda F(u) = \mathcal{K}_u(t^-(u)) \text{ and } \mathcal{K}_u'(t^-(u)) < 0 < \mathcal{K}_u'(t^+(u)).$$

Hence Ψ_u has two critical point corresponding to local minima and local maxima respectively. Therefore, $t^+(u)u \in \mathcal{N}_\lambda^+$, $t^-(u)u \in \mathcal{N}_\lambda^-$. Moreover, $\Psi_u'(t) < 0$ if $t \in (0, t^+(u))$, $\Psi_u'(t) > 0$ for $t \in (t^+, t^-)$ and $\Psi_u'(t) < 0$ for $t \in (t^-(u), \infty)$. Hence, we have

$$\mathcal{I}_\lambda(t^+(u)u) = \inf_{0 \le t \le t_{\max}} \mathcal{I}_\lambda(tu);\ \mathcal{I}_\lambda(t^-(u)u) = \sup_{t \ge t_{\max}} \mathcal{I}_\lambda(tu).$$

This completes the proof. □

We define

$$\Upsilon_1 := \left[\left(\frac{p-r}{2q-r}\right)\|g^+\|_\infty^{-2}(C(N,\alpha))^{-1}S^{\frac{2q}{p}}\right]^{\frac{p-r}{2q-p}}\left(\frac{2q-p}{2q-r}\right)\|f\|_\beta^{-1}S^{\frac{r}{p}}. \tag{4.3.17}$$

Lemma 4.3.5. *There exists $\Upsilon_1 > 0$, defined in* (4.3.17) *such that for all* $0 < \lambda < \Upsilon_1$, *we have $\mathcal{N}_\lambda^0 = \phi$.*

Proof. We will prove it by contradiction. Suppose there exists $\lambda \in \mathbb{R}^+$ with $0 < \lambda < \Upsilon_1$ such that $\mathcal{N}_\lambda^0 \neq \phi$. Then for $u \in \mathcal{N}_\lambda^0$, by (4.3.16), we have

$$\|u\|^p = \lambda\frac{2q-r}{2q-p}\int_\Omega f(x)|u|^r dx \tag{4.3.18}$$

and

$$\|u\|^p = \frac{2q-r}{p-r}\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^q|u(y)|^q}{|x-y|^\alpha}dxdy. \tag{4.3.19}$$

Now, Hölder's inequality and Sobolev embedding theorem in (4.3.18) yield

$$\|u\| \le \left(\lambda\frac{2q-r}{2q-p}S^{-\frac{r}{2}}\|f\|_\beta\right)^{\frac{1}{p-r}}. \tag{4.3.20}$$

and (4.1.1), Hölder's inequality and Sobolev embedding theorem in (4.3.19), we have

$$\|u\|^p \le \frac{2q-r}{p-r}\|g^+\|_\infty^2 C(N,\alpha)S^{-\frac{2q}{p}}\|u\|^{2q}$$

Thus

$$\|u\| \geq \left(\frac{p-r}{2q-r}\|g^+\|_\infty^{-2}(C(N,\alpha))^{-1}S^{\frac{2q}{p}}\right)^{\frac{1}{2q-p}}. \tag{4.3.21}$$

On combining (4.3.20) and (4.3.21), we obtain

$$\lambda \geq \left(\frac{p-r}{2q-r}\|g^+\|_\infty^{-2}(C(N,\alpha))^{-1}S^{\frac{2q}{p}}\right)^{\frac{p-r}{2q-p}} \left(\frac{2q-p}{2q-r}\right)\|f\|_\beta^{-1}S^{\frac{r}{2}} := \Upsilon_1,$$

a contradiction. Hence $\mathcal{N}_\lambda^0 = \phi$. □

Therefore, if $0 < \lambda < \Upsilon_1$, then by Lemma 4.3.5, we can write $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$. Further, we define

$$a_\lambda = \inf_{u\in\mathcal{N}_\lambda} \mathcal{I}_\lambda(u) \ ; \ a_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+} \mathcal{I}_\lambda(u) \ ; \ a_\lambda^- = \inf_{u\in\mathcal{N}_\lambda^-} \mathcal{I}_\lambda(u).$$

By Lemma 4.3.1, we see that $a_\lambda,\ a_\lambda^\pm > -\infty$. The following results are valid.

Lemma 4.3.6. *The following facts hold for same Υ_1 as in* (4.3.17)*:*

- *(i) If $0 < \lambda < \Upsilon_1$, then $a_\lambda \leq a_\lambda^+ < 0$.*
- *(ii) If $0 < \lambda < \frac{r}{p}\Upsilon_1$, then $a_\lambda^- > C_0$, where $C_0 = C_0\,(\lambda, \alpha, r, p, N, S_{H,L}, \|f\|_\beta, \|g^+\|_\infty)$.*

Proof. (*i*) Let $u \in \mathcal{N}_\lambda^+$. Then $\Psi_u''(1) > 0$ implies that

$$\frac{p-r}{2q-r}\|u\|^p > \int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^q|u(y)|^q}{|x-y|^\alpha}dxdy. \tag{4.3.22}$$

On combining (4.2.13), (4.3.14) and (4.3.22), we have

$$\begin{aligned}\mathcal{I}_\lambda(u) &= \left(\frac{1}{p}-\frac{1}{r}\right)\|u\|^p + \left(\frac{1}{r}-\frac{1}{2q}\right)\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^q|u(y)|^q}{|x-y|^\alpha}dxdy \\ &< -\frac{(p-r)(2q-p)}{2prq}\|u\|^p < 0.\end{aligned}$$

Therefore, above conclusion and the definition of a_λ, a_λ^+ yield that $a_\lambda \leq a_\lambda^+ < 0$.

(ii) Let $u \in \mathcal{N}_\lambda^-$. Then $\Psi_u''(1) < 0$. This together with (4.1.1) gives

$$\frac{p-r}{2q-r}\|u\|^p < \int_\Omega \int_\Omega g(x)g(y)\frac{|u(x)|^q|u(y)|^q}{|x-y|^\alpha}dxdy \leq \|g^+\|_\infty^2 C(N,\alpha)S^{-\frac{2q}{p}}\|u\|^{2q}.$$

This implies that

$$\|u\| > \left(\frac{p-r}{2q-r}\|g^+\|_\infty^{-2}(C(N,\alpha))^{-1}S^{\frac{2q}{p}}\right)^{\frac{1}{2q-p}}. \tag{4.3.23}$$

Further, equations (4.3.15) and (4.3.23) yields

$$\begin{aligned}\mathcal{I}_\lambda(u) &\geq \left[\left(\frac{1}{p}-\frac{1}{2q}\right)\|u\|^{p-r} - \lambda\left(\frac{1}{r}-\frac{1}{2q}\right)\|f\|_\beta S_{H,L}^{-\frac{r}{p}}(C(N,\alpha))^{-\frac{r}{2q}}\right]\|u\|^r \\ &> \left[\frac{2q-p}{2pq}\left(\frac{p-r}{2q-r}\|g^+\|_\infty^{-2}C(N,\alpha)^{-1}S^{\frac{2q}{p}}\right)^{\frac{p-r}{2q-r}} - \frac{2q-r}{2rq}\lambda\|f\|_\beta\frac{C(N,\alpha)^{\frac{-r}{2q}}}{S_{H,L}^{\frac{r}{p}}}\right] \\ &\quad\times\left(\frac{p-r}{2q-r}\|g^+\|_\infty^{-2}(C(N,\alpha))^{-1}S^{\frac{2q}{p}}\right)^{\frac{r}{2q-p}}.\end{aligned}$$

Therefore, if $0 < \lambda < \frac{r}{p}\Upsilon_1$, then $\mathcal{I}_\lambda(u) > C_0$ for all $u \in \mathcal{N}_\lambda^-$, where $C_0 = C_0\left(\lambda,\alpha,r,p,N,S_{H,L}, \|f\|_\beta, \|g^+\|_\infty\right)$. □

Lemma 4.3.7. *Let* $0 < \lambda < \Upsilon_1$ *and* Υ_1 *be as in* (4.3.17). *Then for every* $u \in \mathcal{N}_\lambda$, *there exists* $\epsilon > 0$ *and a differentiable function* $\Im : B(0,\epsilon) \subset W_0^{2,p}(\Omega) \to \mathbb{R}^+$ *such that* $\Im(0) = 1$ *and* $\Im(w)(u-w) \in \mathcal{N}_\lambda$ *and for all* $w \in W_0^{2,p}(\Omega)$

$$\langle \Im'(0), w\rangle = \frac{p\int_\Omega |\Delta u|^{p-2}\Delta u\Delta w dx - 2qD(u,w) - r\lambda\int_\Omega f(x)|u|^{r-2}uwdx}{(p-r)\|u\|^p - (2q-r)\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^q|u(y)|^q}{|x-y|^\alpha}dxdy}, \tag{4.3.24}$$

where $D(u,w) = \int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{q-2}u(x)|u(y)|^q w(x)}{|x-y|^\alpha}dxdy$.

Proof. For $u \in \mathcal{N}_\lambda$, define a function $A_u : \mathbb{R}^+ \times W_0^{2,p}(\Omega) \to \mathbb{R}$ such that

$$\begin{aligned}A_u(t,w) &= \langle \mathcal{I}_\lambda'(t(u-w)), t(u-w)\rangle \\ &= t^p\|u-w\|^p - t^r\lambda\int_\Omega f(x)|u-w|^r - t^{2q}\int_\Omega\int_\Omega \frac{g(x)g(y)|u(x)-w(x)|^q|u(y)-w(y)|^q}{|x-y|^\alpha}.\end{aligned}$$

Then $A_u(1,0) = \langle \mathcal{I}'_\lambda(u), u\rangle$ and

$$\frac{d}{dt}A_u(1,0) = p\|u\|^p - \lambda r\int_\Omega f(x)|u(x)|^r dx - 2q\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^q|u(y)|^q}{|x-y|^\alpha}dxdy$$

$$= (p-r)\|u\|^p - (2q-r)\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^q|u(y)|^q}{|x-y|^\alpha}dxdy \neq 0.$$

Now, using the same idea as used in Lemma 2.3.7 of chapter 2, we get the desirable result. □

Lemma 4.3.8. *Let $0 < \lambda < \Upsilon_1$ and Υ_1 be as in (4.3.17). Then for every $u \in \mathcal{N}_\lambda^-$, there exists $\epsilon > 0$ and a differentiable function $\Im^- : B(0,\epsilon) \subset W_0^{2,p}(\Omega) \to \mathbb{R}^+$ such that $\Im^-(0) = 1$ and $\Im^-(w)(u-w) \in \mathcal{N}_\lambda$ and for all $w \in W_0^{2,p}(\Omega)$*

$$\langle (\Im^-)'(0), w\rangle = \frac{p\int_\Omega |\Delta u|^{p-2}\Delta u\Delta w dx - 2qD(u,w) - r\lambda\int_\Omega f(x)|u|^{r-2}uwdx}{(p-r)\|u\|^p - (2q-r)\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^q|u(y)|^q}{|x-y|^\alpha}dxdy},$$

where $D(u,w) = \int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^{q-2}u(x)|u(y)|^q w(x)}{|x-y|^\alpha}dxdy$.

Proof. Proof follows in the similar manner as done in Lemma 4.3.7. □

Lemma 4.3.9. *The following results hold:*

(i) *If $0 < \lambda < \Upsilon_1$, then there exists a sequence $\{u_n\} \subset \mathcal{N}_\lambda$ such that*

$$\mathcal{I}_\lambda(u_n) = a_\lambda + o_n(1) \text{ and } \mathcal{I}'_\lambda(u_n) = o_n(1).$$

(ii) *If $0 < \lambda < \frac{r}{p}\Upsilon_1$, then there exists a sequence $\{u_n\} \subset \mathcal{N}_\lambda^-$ such that*

$$\mathcal{I}_\lambda(u_n) = a_\lambda^- + o_n(1) \text{ and } \mathcal{I}'_\lambda(u_n) = o_n(1).$$

Proof. Ekeland variational principle [27] and Lemma 4.3.1 provide us the existence of a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda$ such that

$$\begin{aligned} &\mathcal{I}_\lambda(u_n) < a_\lambda + \frac{1}{n}, \\ &\mathcal{I}_\lambda(u_n) < \mathcal{I}_\lambda(u) + \frac{1}{n}\|u - u_n\|, \text{ for each } u \in \mathcal{N}_\lambda. \end{aligned} \tag{4.3.25}$$

On taking n large with (4.3.25) together with Lemma 4.3.6 (i), we get

$$\mathcal{I}_\lambda(u_n) = \left(\frac{1}{p} - \frac{1}{2q}\right)\|u_n\|^p - \lambda\left(\frac{1}{r} - \frac{1}{2q}\right)\int_\Omega f(x)|u_n|^r < a_\lambda + \frac{1}{n} < \frac{a_\lambda}{2}. \tag{4.3.26}$$

Thus

$$-\lambda\left(\frac{1}{r} - \frac{1}{2q}\right)\int_\Omega f(x)|u_n|^r dx \le \mathcal{I}_\lambda(u_n) < \frac{a_\lambda}{2} < 0,$$

and it follows

$$0 < -\frac{2qra_\lambda}{2(2q-r)} < \lambda\int_\Omega f(x)|u_n|^r dx \le \lambda S^{-\frac{r}{p}}\|f\|_\beta\|u_n\|^r. \tag{4.3.27}$$

Therefore, $u_n \neq 0$. Now using (4.3.26) and Hölder's inequality, we have

$$\|u_n\| < \left(\frac{2\lambda(2q-r)}{r(2q-p)}S^{-\frac{r}{p}}\|f\|_\beta\right)^{\frac{1}{p-r}} \tag{4.3.28}$$

and equation (4.3.27) gives

$$\|u_n\| > \left(\frac{-2qra_\lambda}{2\lambda(2q-r)}S^{\frac{r}{p}}\|f\|_\beta^{-1}\right)^{\frac{1}{r}}. \tag{4.3.29}$$

Further, we show that $\|\mathcal{I}'_\lambda(u_n)\|_{W_0^{2,p}(\Omega)^{-1}} \to 0$ as $n \to \infty$. Applying Lemma 4.3.7 to u_n, we obtain a sequence of functions $\Im_n : B(0, \epsilon_n) \to \mathbb{R}^+$ for some $\epsilon_n > 0$ such that $\Im_n(w)(u_n - w) \in \mathcal{N}_\lambda$. Choose $0 < \vartheta < \epsilon_n$. Let $0 \neq u \in W_0^{2,p}(\Omega)$ and take $w_\vartheta^* = \frac{\vartheta u}{\|u\|}$ and $w_\vartheta = \zeta_n(w_\vartheta^*)(u_n - w_\vartheta^*)$. Since $w_\vartheta \in \mathcal{N}_\lambda$, equation (4.3.25) implies that

$$\mathcal{I}_\lambda(w_\vartheta) - \mathcal{I}_\lambda(u_n) \ge -\frac{1}{n}\|w_\vartheta - u_n\|.$$

By mean value theorem, we have

$$\langle \mathcal{I}_\lambda'(u_n), w_\vartheta - u_n\rangle + o(\|w_\vartheta - u_n\|) \geq -\frac{1}{n}\|w_\vartheta - u_n\|.$$

Therefore

$$\langle \mathcal{I}_\lambda'(u_n), -w_\vartheta^*\rangle + (\Im_n(w_\vartheta^*) - 1)\,\langle \mathcal{I}_\lambda'(u_n), (u_n - w_\vartheta^*)\rangle \geq -\frac{1}{n}\|w_\vartheta - u_n\| + o(\|w_\vartheta - u_n\|). \tag{4.3.30}$$

Since $\Im_n(w_\vartheta^*)(u_n - w_\vartheta^*) \in \mathcal{N}_\lambda$ and from (4.3.30), we get

$$-\vartheta\left\langle \mathcal{I}_\lambda'(u_n), \frac{u}{\|u\|}\right\rangle + (\Im_n(w_\vartheta^*) - 1)\langle \mathcal{I}_\lambda'(u_n) - \mathcal{I}_\lambda'(w_\vartheta), u_n - w_\vartheta^*\rangle \geq -\frac{1}{n}\|w_\vartheta - u_n\| + o(\|w_\vartheta - u_n\|).$$

Hence

$$\begin{aligned}\left\langle \mathcal{I}_\lambda'(u_n), \frac{u}{\|u\|}\right\rangle \leq & \frac{1}{n\vartheta}\|w_\vartheta - u_n\| + \frac{1}{\vartheta}o(\|w_\vartheta - u_n\|)\\ & + \frac{(\Im_n(w_\vartheta^*) - 1)}{\vartheta}\langle \mathcal{I}_\lambda'(u_n - w_\vartheta), u_n - w_\vartheta^*\rangle. \end{aligned} \tag{4.3.31}$$

By virtue of $\|w_\vartheta - u_n\| \leq \vartheta|\Im_n(w_\vartheta^*)| + |\Im_n(w_\vartheta^*) - 1|\|u_n\|$ and $\lim_{\vartheta\to 0}\frac{|\Im_n(w_\vartheta^*)-1|}{\vartheta} \leq \|\Im_n'(0)\|$, if we take $\vartheta \to 0$ in (4.3.31) for a fixed $n \in \mathbb{N}$ and using (4.3.29), we can find a constant $M > 0$, independent from ϑ such that

$$\left\langle \mathcal{I}_\lambda'(u_n), \frac{u}{\|u\|}\right\rangle \leq \frac{M}{n}(1 + \|\Im_n'(0)\|).$$

Further, we show that $\|\Im_n'(0)\|$ is uniformly bounded.
From (4.3.24), (4.3.28), Hardy-Littlewood-Sobolev inequality, and Sobolev embedding, we have

$$|\langle \Im_n'(0), w\rangle| \leq \frac{M_1\|w\|}{\left|(p-r)\|u_n\|^p - (2q-r)\int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^q|u_n(y)|^q}{|x-y|^\alpha}dxdy\right|},$$

for some $M_1 > 0$. Now, we only need to show that

$$\left|(p-r)\|u_n\|^p - (2q-r)\int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^q|u_n(y)|^q}{|x-y|^\alpha}dxdy\right| > d,$$

for some $d > 0$ and n large enough. On contrary, assume that there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ such that

$$(p-r)\|u_n\|^p - (2q-r)\int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^q|u_n(y)|^q}{|x-y|^\alpha}dxdy = o_n(1). \tag{4.3.32}$$

On combining (4.3.29) and (4.3.32), we see, there exists a constant $s > 0$ such that

$$\int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^q|u_n(y)|^q}{|x-y|^\alpha}dxdy \geq s,$$

for n sufficiently large. On using $u_n \in \mathcal{N}_\lambda$ with (4.3.28) and (4.3.32), we have

$$\begin{aligned}\lambda\int_\Omega f(x)|u_n|^r dx &= \|u_n\|^p - \int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^q|u_n(y)|^q}{|x-y|^\alpha}dxdy \\ &= \frac{2q-p}{p-r}\int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^q|u_n(y)|^q}{|x-y|^\alpha}dxdy + o_n(1).\end{aligned} \tag{4.3.33}$$

Define $E_\lambda : \mathcal{N}_\lambda \to \mathbb{R}$ such that

$$E_\lambda(u) = C\left(\frac{\|u\|^{p(2q-p+1)}}{\int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^q|u(y)|^q}{|x-y|^\alpha}dxdy}\right)^{\frac{1}{2q-p}} - \lambda\int_\Omega f(x)|u|^r dx, \tag{4.3.34}$$

where $C = \left(\frac{2q-p}{p-r}\right)\left(\frac{p-r}{2q-r}\right)^{\frac{2q-p+1}{2q-p}}$. As $u_n \in \mathcal{N}_\lambda$ so using (4.3.33) and (4.3.32) in (4.3.34), we obtain

$$\begin{aligned}E_\lambda(u_n) = C&\left[\frac{\left(\frac{2q-r}{p-r}\right)^{2q-p+1}\left(\int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^q|u_n(y)|^q}{|x-y|^\alpha}dxdy\right)^{2q-p+1}}{\int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^q|u_n(y)|^q}{|x-y|^\alpha}dxdy}\right]^{\frac{1}{2q-p}} \\ &- \frac{2q-p}{p-r}\int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^q|u_n(y)|^q}{|x-y|^\alpha}dxdy + o_n(1)\end{aligned}$$

$$
\begin{aligned}
&= \left[\left(\frac{2q-p}{p-r} \right) \left(\frac{p-r}{2q-r} \right)^{\frac{2q-p+1}{2q-p}} \left(\frac{2q-r}{p-r} \right)^{\frac{2q-p+1}{2q-p}} - \frac{2q-p}{p-r} \right] \\
&\quad \times \int_\Omega \int_\Omega g(x)g(y) \frac{|u_n(x)|^q |u_n(y)|^q}{|x-y|^\alpha} dxdy + o_n(1) \\
&= o_n(1).
\end{aligned}
$$

However, from (4.1.2), (4.3.20) and $\lambda \in (0, \Upsilon_1)$, we have

$$
\begin{aligned}
E_\lambda(u_n) &\geq C \left(\frac{\|u_n\|^{p(2q-p+1)}}{\|u_n\|^{2q}} \|g^+\|_\infty^{-2} S_{H,L}^{\frac{2q}{p}} \right)^{\frac{1}{2q-p}} - \lambda \|f\|_\beta S^{-\frac{r}{p}} \|u_n\|^r \\
&= \|u_n\|^r \left(C \|u_n\|^{p-1-r} \|g^+\|_\infty^{-\frac{2}{2q-p}} S_{H,L}^{\frac{2q}{2q-p}} - \lambda \|f\|_\beta S^{-\frac{r}{p}} \right) \\
&\geq \|u_n\|^r \left[\left(\frac{2q-p}{p-r} \right) \left(\frac{p-r}{2q-r} \right)^{\frac{2q-p+1}{2q-p}} \|g^+\|_\infty^{-\frac{2}{2q-p}} S_{H,L}^{\frac{2q}{p(2q-p)}} \right. \\
&\qquad \left. \left(\lambda \frac{2q-r}{2q-p} \|f\|_\beta S^{-\frac{r}{p}} \right)^{\frac{p-1-r}{p-r}} - \lambda \|f\|_\beta S^{-\frac{r}{p}} \right] \\
&= \|u_n\|^r \left(\lambda^{\frac{p-1-r}{p-r}} \|f\|_\beta S^{-\frac{r}{p}} \Upsilon_1^{\frac{1}{p-r}} - \lambda \|f\|_\beta S^{-\frac{r}{q}} \right) \\
&= \|u_n\|^r \lambda^{\frac{p-1-r}{p-r}} \|f\|_\beta S^{-\frac{r}{p}} \left(\Upsilon_1^{\frac{1}{p-r}} - \lambda^{\frac{1}{p-r}} \right) > 0,
\end{aligned}
$$

which is a contradiction. Hence $\left\langle \mathcal{I}'_\lambda(u_n), \frac{u}{\|u\|} \right\rangle \leq \frac{c}{n}$. This completes the proof.
(ii) Using Lemma 4.3.8, the proof follows same as given in (i). $\square$

Lemma 4.3.10. *Suppose $\{u_n\} \subset W_0^{2,p}(\Omega)$ is a $(PS)_c$-sequence for $\mathcal{I}_\lambda$, then $\{u_n\}$ is bounded in $W_0^{2,p}(\Omega)$.*

Proof. Let $\{u_n\} \subset W_0^{2,p}(\Omega)$ be a $(PS)_c$-sequence for $\mathcal{I}_\lambda$, then by the definition of $(PS)_c$-sequence $\mathcal{I}_\lambda(u_n) \to c$ and $\mathcal{I}'_\lambda(u_n) \to 0$ in $(W_0^{2,p}(\Omega))^{-1}$. Thus, we have

$$
\frac{1}{p}\|u_n\|^p - \frac{1}{2q} \int_\Omega \int_\Omega g(x)g(y) \frac{|u_n(x)|^q |u_n(y)|^q}{|x-y|^\alpha} - \frac{\lambda}{r} \int_\Omega f(x)|u_n|^r dx = c + o_n(1). \quad (4.3.35)
$$

$$
\|u_n\|^p - \int_\Omega \int_\Omega g(x)g(y) \frac{|u_n(x)|^q |u_n(y)|^q}{|x-y|^\alpha} - \lambda \int_\Omega f(x)|u_n|^r dx = o_n(1). \quad (4.3.36)
$$

Our aim is to show that $\{u_n\}$ is bounded. On contrary, suppose $\|u_n\| \to \infty$, then define $\overline{u}_n := \frac{u_n}{\|u_n\|}$ with $\|\overline{u}_n\| = 1$. This implies $\{\overline{u}_n\}$ is a bounded sequence. So, up to subsequence, $u_n \rightharpoonup \overline{u}$ weakly in $W_0^{2,p}(\Omega)$, $\overline{u}_n \to \overline{u}$ strongly in $L^m(\Omega)$, where $1 \leq m < p^*$, and $\overline{u}_n(x) \to \overline{u}(x)$ pointwise a.e. in Ω. Using (4.3.35) and (4.3.36), we obtain

$$\frac{\|\overline{u}_n\|^p}{p} - \frac{\|u_n\|^{2q-p}}{2q}\int_\Omega\int_\Omega g(x)g(y)\frac{|\overline{u}_n(x)|^q|\overline{u}_n(y)|^q}{|x-y|^\alpha}dxdy - \frac{\lambda\|u_n\|^{r-p}}{r}\int_\Omega f|\overline{u}_n|^r dx = o_n(1). \tag{4.3.37}$$

$$\|\overline{u}_n\|^p - \|u_n\|^{2q-p}\int_\Omega\int_\Omega g(x)g(y)\frac{|\overline{u}_n(x)|^q|\overline{u}_n(y)|^q}{|x-y|^\alpha}dxdy - \lambda\|u_n\|^{r-p}\int_\Omega f|\overline{u}_n|^r dx = o_n(1). \tag{4.3.38}$$

Using (4.3.37) in (4.3.38), we get

$$\|\overline{u}_n\|^p = \frac{2q}{p}\|\overline{u}_n\|^p + \frac{r-2q}{r}\lambda\|u_n\|^{r-p}\int_\Omega f(x)|\overline{u}_n|^r dx + o_n(1).$$

This implies that

$$\|\overline{u}_n\|^p = \frac{p(2q-r)}{r(2q-p)}\lambda\|u_n\|^{r-p}\int_\Omega f(x)|\overline{u}|^r dx + o_n(1) \to 0 \ \text{ as } \ n \to \infty,$$

since $\|u_n\| \to \infty$ and $r < p$. Which gives a contradiction as $\|\overline{u}_n\| = 1$ and the result follows. □

4.4 Multiplicity results in subcritical case

Lemma 4.4.1. *Suppose that assumptions (f_1) and (g_1) hold and $\frac{p(2N-\alpha)}{2N} \leq q < p_\alpha^*$, then $\mathcal{I}_\lambda$ satisfies the $(PS)_c$ condition i.e. if $\{u_n\}$ is a sequence in $W_0^{2,p}(\Omega)$ satisfying*

$$\mathcal{I}_\lambda(u_n) \to c \ \text{ and } \ \mathcal{I}'_\lambda(u_n) \to 0 \ \text{ as } \ n \to \infty, \tag{4.4.39}$$

then $\{u_n\}$ has a convergent subsequence.

Proof. Suppose $\{u_n\}$ is a sequence which satisfies (4.4.39). Then by Lemma (4.3.10), $\{u_n\}$ is bounded. So up to subsequence, there exists $u \in W_0^{2,p}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{2,p}(\Omega)$, $u_n \to u$ strongly in $L^m(\Omega)$ for all $1 \leq m < p^*$ and $u_n(x) \to u(x)$ pointwise a.e. in Ω. Furthermore,

$$\begin{aligned}\langle \mathcal{I}'_\lambda(u_n), u_n - u\rangle &= \int_\Omega |\Delta u_n|^{p-2}\Delta u_n \Delta(u_n - u)dx - \lambda \int_\Omega f(x)|u_n|^{r-2}u_n(u_n - u)dx \\ &\quad - \int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^q|u_n(y)|^{q-2}u_n(y)(u_n(y) - u(y))}{|x-y|^\alpha}dxdy, \\ &\to 0 \text{ as } n \to \infty. \qquad (4.4.40)\end{aligned}$$

Also

$$\begin{aligned}&\int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^q|u_n(y)|^{q-2}u_n(y)(u_n(y) - u(y))}{|x-y|^\alpha}dxdy \\ &\leq \|g^+\|_\infty^2 \left(\int_\Omega |u_n|^{q-1}|u_n - u|^{\frac{2N}{2N-\alpha}}\right)^{\frac{2N-\alpha}{2N}} \left(\int_\Omega |u_n|^{\frac{2Nq}{2N-\alpha}}\right)^{\frac{2N-\alpha}{2N}} \\ &\leq \|g^+\|_\infty^2 \left[\left(\int_\Omega |u_n|^{\frac{2Nq}{2N-\alpha}}\right)^{\frac{q-1}{q}} \left(\int_\Omega |u_n - u|^{\frac{2Nq}{2N-\alpha}}\right)^{\frac{1}{q}}\right]^{\frac{2N-\alpha}{2N}} \left(\int_\Omega |u_n|^{\frac{2Nq}{2N-\alpha}}\right)^{\frac{2N-\alpha}{2N}} \\ &\leq \|g^+\|_\infty^2 \left(\int_\Omega |u_n - u|^{\frac{2Nq}{2N-\alpha}}\right)^{\frac{2N-\alpha}{2Nq}} \left(\int_\Omega |u_n|^{\frac{2Nq}{2N-\alpha}}\right)^{\frac{(2N-\alpha)(q-1)}{2Nq}} \\ &\to 0 \text{ as } n \to \infty. \qquad (4.4.41)\end{aligned}$$

As $u_n \to u$ strongly in $L^m(\Omega)$, for $1 \leq m < p^*$, then one can easily calculate

$$\left|\int_\Omega f(x)|u_n|^{r-2}u_n(u_n - u)dx\right| \to 0 \text{ as } n \to \infty.$$

This with (4.4.41) and (4.4.40) yield that

$$\lim_{n\to\infty}\left(\int_\Omega |\Delta u_n|^{p-2}\Delta u_n \Delta(u_n - u)dx\right) = 0.$$

Further, using the following inequality

$$|a-b|^p \leq \begin{cases} C(a-b)\left[(|a|^{p-2}a-|b|^{p-2}b)(a-b)\right]^{\frac{p}{2}}(|a|^p+|b|^p)^{\frac{2-p}{2}}, \text{ for } 1<p<2 \\ C(a-b)(|a|^{p-2}a-|b|^{p-2}b), \text{ for } p\geq 2, \end{cases}$$

and take $a=\Delta u_n$, $b=\Delta u$, we can see that $\lim_{n\to\infty}\int_\Omega |\Delta(u_n-u)|^p dx = 0$, $\forall p \geq 1$. This implies that $u_n \to u$ strongly in $W_0^{2,p}(\Omega)$ and this completes the proof.

□

Lemma 4.4.2. *Let* $0<\lambda<\Upsilon_1$ *and* Υ_1 *be same as in* (4.3.17). *If* $1<r<p$ *and* $\frac{p(2N-\alpha)}{2N} \leq q < p_\alpha^*$, *then* $\mathcal{I}_\lambda$ *has a minimizer* u_1 *in* $\mathcal{N}_\lambda^+$ *satisfying the following:*

(*i*) $\mathcal{I}_\lambda(u_1)=a_\lambda=a_\lambda^+<0$;

(*ii*) u_1 *is a nontrivial solution;*

(*iii*) $\mathcal{I}_\lambda(u_1)\to 0$ *as* $\lambda\to 0^+$.

Proof. By Lemma 4.3.9 (*i*) and Lemma 4.3.6, we see that there exists a $(PS)_c$-sequence $\{u_n\}\in\mathcal{N}_\lambda$ for $\mathcal{I}_\lambda$ such that

$$\lim_{n\to\infty}\mathcal{I}_\lambda(u_n)=a_\lambda\leq a_\lambda^+<0 \quad \text{and} \quad \lim_{n\to\infty}\mathcal{I}_\lambda'(u_n)=0 \text{ in } X^{-1}.$$

Further, by Lemma 4.4.1, we infer that $u_n\to u_1$ strongly in $W_0^{2,p}(\Omega)$. Thus

$$\lim_{n\to\infty}\mathcal{I}_\lambda(u_n)=\mathcal{I}_\lambda(u_1)=a_\lambda.$$

Therefore u_1 is a minimizer of $\mathcal{I}_\lambda$. Further we claim that $u_1\in\mathcal{N}_\lambda^+$. Supposing the contrary, $u_1\in\mathcal{N}_\lambda^-$, then by Lemma 4.3.4 (*iv*), there exists $t^+<t^-=1$ such that $t^+u_1\in\mathcal{N}_\lambda^+$. Since $\Psi_{u_1}'(t^+)=0$ and $\Psi_{u_1}''(t^+)>0$, there exists t^* satisfying $t^+<t^*\leq t^-$ such that $\mathcal{I}_\lambda(t^+u_1)<\mathcal{I}_\lambda(t^*u_1)$. Thus

$$a_\lambda\leq\mathcal{I}_\lambda(t^+u_1)<\mathcal{I}_\lambda(t^*u_1)\leq\mathcal{I}_\lambda(t^-u_1)=\mathcal{I}_\lambda(u_1)=a_\lambda,$$

which is a contradiction. Hence $u_1\in\mathcal{N}_\lambda^+$.

(iii) Further, from Lemma 4.3.6 (i) and (4.3.15), we obtain

$$0 > a_\lambda^+ \geq a_\lambda = \mathcal{I}_\lambda(u_1) > -\frac{2q-r}{2qr}\lambda\|f\|_\beta S_{H,L}^{-\frac{r}{p}}(C(N,\alpha))^{-\frac{r}{2q}}\|u_1\|^r,$$

which gives us that $\mathcal{I}_\lambda(u_1) \to 0$ as $\lambda \to 0^+$. $\square$

Lemma 4.4.3. *Assume that* $(f1)$, $(g1)$ *hold. If* $1 < r < p$ *and* $\frac{p(2N-\alpha)}{2N} \leq q < p_\alpha^*$, *then there exists* $\Upsilon_2 > 0$ *for* $0 < \lambda < \Upsilon_2$ *such that* $\mathcal{I}_\lambda$ *has a minimizer* $u_2 \in \mathcal{N}_\lambda^-$ *with*
(i) $\mathcal{I}_\lambda(u_2) = a_\lambda^-$;
(ii) u_2 *is a nontrivial solution,*
where $\Upsilon_2 = \min\{\Upsilon_1, \frac{p}{r}\Upsilon_1\}$.

Proof. By Lemma 4.3.9 (ii), there exists a $(PS)_{a_\lambda^-}$- sequence $\{u_n\} \subset \mathcal{N}_\lambda^-$ such that

$$\lim_{n\to\infty} \mathcal{I}_\lambda(u_n) = a_\lambda^- \text{ and } \lim_{n\to\infty} \mathcal{I}'_\lambda(u_n) = 0 \text{ in } X^{-1}.$$

On using Lemma 4.4.1, we see that there exists a subsequence denoted same as $\{u_n\}$ and $u_2 \in \mathcal{N}_\lambda^-$ such that $u_n \to u_2$ in $W_0^{2,p}(\Omega)$ and for $0 < \lambda < \Upsilon_2$, $\mathcal{I}_\lambda(u_2) = a_\lambda^- > 0$. Therefore, by the same arguments used in Lemma 4.4.2, for $0 < \lambda < \Upsilon_2$, u_2 is a nontrivial solution. $\square$

Proof of Theorem 4.0.1 and Theorem 4.0.2: By Lemma 4.4.2 and Lemma 4.4.3, we find the existence of two nontrivial solution $u_1 \in \mathcal{N}_\lambda^+$ and $u_2 \in \mathcal{N}_\lambda^-$ respectively. As $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \phi$, this implies u_1 and u_2 are distinct. This completes the proof. $\square$

4.5 Existence of a solution in critical case

Next we show the existence of a nontrivial solution in critical case $q = p_\alpha^*$. For this, we show that $\mathcal{I}_\lambda$ satisfies the Palais-Smale condition by using the following Lemma.

Lemma 4.5.1. *Suppose assumptions* $(f1)$, $(g1)$ *hold and* $\{u_n\} \subset W_0^{2,p}(\Omega)$ *is a* $(PS)_c$*-sequence for* $\mathcal{I}_\lambda$ *such that* $u_n \rightharpoonup u$ *weakly in* $W_0^{2,p}(\Omega)$. *Then* $\mathcal{I}'_\lambda(u) = 0$ *and* $\mathcal{I}_\lambda(u) \geq -Q_0\lambda^{\frac{p}{p-r}}$, *where* Q_0 *is a positive constant depending on* N, p, α, r, S *and* $\|f\|_\beta$ *respectively.*

Proof. Since $\{u_n\}$ is a $(PS)_c$-sequence for $\mathcal{I}_\lambda$ with $u_n \rightharpoonup u$ weakly in $W_0^{2,p}(\Omega)$. Then by standard arguments, we have $\mathcal{I}'_\lambda(u) = 0$ or $\langle \mathcal{I}'_\lambda(u), u\rangle = 0$. This implies that

$$\|u\|^p - \int_\Omega\int_\Omega g(x)g(y)\frac{|u(x)|^q|u(y)|^q}{|x-y|^\alpha}dxdy - \lambda\int_\Omega f(x)|u|^r dx = 0.$$

Thus, we have

$$\mathcal{I}_\lambda(u) = \left(\frac{1}{p} - \frac{1}{2q}\right)\|u\|^p - \lambda\left(\frac{1}{r} - \frac{1}{2q}\right)\int_\Omega f(x)|u|^r dx.$$

On applying Hölder's inequality, Sobolev embedding theorem and Young's inequality, we obtain

$$\begin{aligned}\mathcal{I}_\lambda(u) &\geq \left(\frac{1}{p} - \frac{1}{2q}\right)\|u\|^p - \lambda\left(\frac{1}{r} - \frac{1}{2q}\right)S^{-\frac{r}{p}}\|f\|_\beta\|u\|^r \\ &\geq \left(\frac{2q-p}{2qp}\right)\|u\|^p - \left(\frac{2q-r}{2qr}\right)S^{-\frac{r}{p}}\left[\frac{r}{p}\varepsilon^{-\frac{p}{r}}\|u\|^p + \left(\frac{p-r}{p}\right)\varepsilon^{\frac{p}{p-r}}\left(\lambda\|f\|_\beta\right)^{\frac{p}{p-r}}\right] \\ &= -\,Q_0\lambda^{\frac{p}{p-r}},\end{aligned}$$

where $Q_0 = \left(\frac{2q-r}{2qr}\right)\left(\frac{p-r}{p}\right)S^{-\frac{r}{p}}\left(\varepsilon\|f\|_\beta\right)^{\frac{p}{p-r}}$ with $\varepsilon = \left[\left(\frac{2q-r}{2q-p}\right)S^{-\frac{r}{p}}\right]^{\frac{r}{p}}$. $\square$

Lemma 4.5.2. *Suppose that assumptions (f_1) and (g_1) hold, then $\mathcal{I}_\lambda$ satisfies the $(PS)_c$-condition for all c in the interval*

$$-\infty < c < \frac{N+2p-\alpha}{p(2N-\alpha)}\|g^+\|_\infty^{-\frac{2p}{2p^*_\alpha - p}} S_{H,L}^{\frac{2p^*_\alpha}{2p^*_\alpha - p}} - Q_0\lambda^{\frac{p}{p-r}} := c_\infty,$$

where Q_0 is defined in Lemma 4.5.1.

Proof. Let $\{u_n\} \subset W_0^{2,p}(\Omega)$ be a $(PS)_c$-sequence for $\mathcal{I}_\lambda$ with $c \in (-\infty, c_\infty)$. Then Lemma 4.3.10 implies that $\{u_n\}$ is bounded in $W_0^{2,p}(\Omega)$. So up to a subsequence, there exists $u \in W_0^{2,p}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{2,p}(\Omega)$, $u_n \to u$ strongly in $L^m(\Omega)$ for all $1 \leq m < p^*$ and $u_n(x) \to u(x)$ pointwise a.e. in Ω. Also, we have $|\Delta u_n|^p \rightharpoonup \mu$, $|u_n|^{p^*} \rightharpoonup \omega$, $\left(\int_\Omega \frac{|u_n(y)|^{p^*_\alpha}}{|x-y|^\alpha}dy\right)|u_n(x)|^{p^*_\alpha} \rightharpoonup \eta$ in the sense of measures, where μ, ω and η are bounded and nonnegative measures on $\mathbb{R}^N$. Then by applying Lemma 4.2.2 there exists a sequence of points $\{x_i\}_{i\in J} \subset \mathbb{R}^N$ and families of positive numbers $\{\mu_i : i \in J\}$, $\{\omega_i : i \in J\}$ and $\{\eta_i : i \in J\}$ such that

$$\begin{aligned}\eta &= \left(\int_\Omega \frac{|u(y)|^{p^*_\alpha}}{|x-y|^\alpha}dy\right)|u(x)|^{p^*_\alpha} + \sum_{i\in J}\eta_i\delta_{x_i},\\ \mu &\geq |\Delta u|^p + \sum_{i\in J}\mu_i\delta_{x_i},\ \omega \geq |u|^{p^*} + \sum_{i\in J}\omega_i\delta_{x_i}, \\ S_{H,L}\eta_i^{\frac{N-2p}{2N-\alpha}} &\leq \mu_i,\ \eta_i \leq C(N,\alpha)\omega_i^{\frac{2N-\alpha}{N}},\end{aligned} \tag{4.5.42}$$

where J is an at most countable set and δ_{x_i} is the Dirac mass at x_i.
Now, we claim that $J = \phi$. Define a fixed cut-off function $\phi_\epsilon \in C_c^\infty(\Omega)$ such that $0 \leq \phi_\epsilon \leq 1$ in $\mathbb{R}^N$, $\phi_\epsilon \equiv 1$ on $B(x_i,\epsilon)$, $\phi_\epsilon \equiv 0$ on $\mathbb{R}^N \setminus B(x_i, 2\epsilon)$ with $|\nabla\phi_\epsilon| \leq \frac{2C}{\epsilon}$, $|\Delta\phi_\epsilon| \leq \frac{2C}{\epsilon^2}$. Since the sequence $\{\phi_\epsilon u_n\}$ is bounded in $W_0^{2,p}(\Omega)$, then

$$\lim_{n\to\infty}\langle \mathcal{I}'_\lambda(u_n), \phi_\epsilon u_n\rangle = 0,$$

i.e.

$$\lim_{n\to\infty}\left(\int_\Omega |\Delta u_n|^{p-2}\Delta u_n \Delta(\phi_\epsilon u_n)dx - \int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^{p^*_\alpha}|u_n(y)|^{p^*_\alpha}\phi_\epsilon}{|x-y|^\alpha}dxdy \right. \\ \left. -\lambda\int_\Omega f(x)|u_n|^r\phi_\epsilon dx\right) = 0. \tag{4.5.43}$$

We know that

$$\int_\Omega |\Delta u_n|^{p-2}\Delta u_n\Delta(\phi_\epsilon u_n)dx = \int_\Omega |\Delta u_n|^{p-2}\Delta u_n\left(\Delta\phi_\epsilon u_n + 2\nabla\phi_\epsilon.\nabla u_n + \phi_\epsilon\Delta u_n\right)dx.$$

Consider

$$\begin{aligned}0 &\leq \lim_{n\to\infty}\left|\int_\Omega |\Delta u_n|^{p-2}\Delta u_n(\nabla\phi_\epsilon.\nabla u_n)dx\right| \leq \lim_{n\to\infty}\left|\int_\Omega |\Delta u_n|^{p-1}|\nabla\phi_\epsilon||\nabla u_n|dx\right|\\ &\leq \lim_{n\to\infty}\left(\int_\Omega |\Delta u_n|^p dx\right)^{\frac{p-1}{p}}\left(\int_\Omega |\nabla\phi_\epsilon|^p|\nabla u_n|^p dx\right)^{\frac{1}{p}}\\ &\leq C\left(\int_{B(x_i,2\epsilon)} |\nabla\phi_\epsilon|^p|\nabla u_n|^p dx\right)^{\frac{1}{p}}\end{aligned}$$

$$
\begin{aligned}
\leq & C\left(\int_{B(x_i,2\epsilon)}|\nabla\phi_\epsilon|^N dx\right)^{\frac{1}{N}}\left(\int_{B(x_i,2\epsilon)}|\nabla u|^{\frac{Np}{N-p}}dx\right)^{\frac{N-p}{Np}}\\
\leq & C'\left(\int_{B(x_i,2\epsilon)}|\nabla u|^{\frac{Np}{N-p}}dx\right)^{\frac{N-p}{Np}} \longrightarrow 0 \text{ as } \epsilon\longrightarrow 0,
\end{aligned}
$$

Thus using the same idea, we obtain

$$
\begin{aligned}
0 \leq \lim_{n\to\infty}&\left|\int_\Omega |\Delta u_n|^{p-2}\Delta u_n(u_n\Delta\phi_\epsilon)\right| \leq \lim_{n\to\infty}\left(\int_\Omega|\Delta u_n|^p\right)^{\frac{p-1}{p}}\left(\int_\Omega|\Delta\phi_\epsilon|^p|u_n|^p\right)^{\frac{1}{p}}\\
\leq & C\left(\int_{B(x_i,2\epsilon)}|\Delta\phi_\epsilon|^{\frac{N}{2}}dx\right)^{\frac{2}{N}}\left(\int_{B(x_i,2\epsilon)}|u|^{p^*}dx\right)^{\frac{1}{p^*}}\\
& \leq C''\left(\int_{B(x_i,2\epsilon)}|u|^{p^*}dx\right)^{\frac{1}{p^*}} \longrightarrow 0 \text{ as } \epsilon\longrightarrow 0.
\end{aligned} \tag{4.5.44}
$$

On combining (4.5.43) and (4.5.44), we have

$$
\begin{aligned}
0 = & \lim_{\epsilon\to 0}\lim_{n\to\infty}\left(\int_\Omega|\Delta u_n|^{p-2}\Delta u_n\phi_\epsilon\Delta u_n - \int_\Omega\int_\Omega g(x)g(y)\frac{|u_n(x)|^{p^*_\alpha}|u_n(y)|^{p^*_\alpha}\phi_\epsilon}{|x-y|^\alpha}dxdy\right.\\
&\qquad\qquad\qquad\qquad\left. -\lambda\int_{B(x_i,2\epsilon)}f(x)|u_n|^r\phi_\epsilon dx\right)\\
\geq & \mu_i - \|g^+\|_\infty^2\eta_i.
\end{aligned}
$$

With the help of relation between $S_{H,L}$, η_i and μ_i given by (4.5.42), we obtain

$$
\left(\mu_i\|g^+\|_\infty^{-2}\right)^{\frac{p}{2p^*_\alpha}} S_{H,L} \leq \mu_i.
$$

This implies that either $\mu_i = 0$ or $\mu_i \geq \|g^+\|_\infty^{-\frac{2p}{2p^*_\alpha-p}} S_{H,L}^{\frac{2p^*_\alpha}{2p^*_\alpha-p}}$. Further, we will show that $\mu_i = 0$ for every i. On contrary, suppose there exists some k such that $\mu_k \neq 0$ and $\mu_k \geq \|g^+\|_\infty^{-\frac{2p}{2p^*_\alpha-p}} S_{H,L}^{\frac{2p^*_\alpha}{2p^*_\alpha-p}}$.
Since $\{u_n\}$ is a $(PS)_c$-sequence, we obtain

$$
\begin{aligned}
c &= \lim_{n\to\infty}\left(\mathcal{I}_\lambda(u_n) - \frac{1}{p}\langle \mathcal{I}'_\lambda(u_n), u_n\rangle\right)\\
&= \lim_{n\to\infty}\left[\left(\frac{1}{p}-\frac{1}{2p^*_\alpha}\right)\|u_n\|^p - \lambda\left(\frac{1}{r}-\frac{1}{2p^*_\alpha}\right)\int_\Omega f(x)|u_n|^r dx\right]\\
&\geq \left(\frac{1}{p}-\frac{1}{2p^*_\alpha}\right)\left(\|u\|^p + \sum_{i\in J}\mu_i\delta_{x_i}\right) - \lambda\left(\frac{1}{r}-\frac{1}{2p^*_\alpha}\right)S^{-\frac{r}{p}}\|f\|_\beta\|u\|^r\\
&\geq \left(\frac{1}{p}-\frac{1}{2p^*_\alpha}\right)(\|u\|^p+\mu_k) - \lambda\left(\frac{1}{r}-\frac{1}{2p^*_\alpha}\right)S^{-\frac{r}{p}}\|f\|_\beta\|u\|^r\\
&\geq \left(\frac{1}{p}-\frac{1}{2p^*_\alpha}\right)\|g^+\|_\infty^{\frac{-2p}{2p^*_\alpha-p}} S_{H,L}^{\frac{2p^*_\alpha}{2p^*_\alpha-p}} + \left(\frac{1}{p}-\frac{1}{2p^*_\alpha}\right)\|u\|^p - \lambda\left(\frac{1}{r}-\frac{1}{2p^*_\alpha}\right)S^{\frac{-r}{p}}\|f\|_\beta\|u\|^r.
\end{aligned}
$$

Suppose $H(t) = at^p - \lambda bt^r$, where $a = \left(\frac{1}{p}-\frac{1}{2p^*_\alpha}\right)$ and $b = \left(\frac{1}{r}-\frac{1}{2p^*_\alpha}\right)S^{-\frac{r}{p}}\|f\|_\beta$. Then it attains its absolute minimum at $t^* = \left(\frac{b\lambda r}{ap}\right)^{\frac{1}{p-r}}$. Thus $H(t)\geq H(t^*) = -Q_0\lambda^{\frac{p}{p-r}}$ for $t>0$, where Q_0 is calculated in Lemma 4.5.1. Therefore

$$
c \geq \frac{N+2p-\alpha}{p(2N-\alpha)}\|g^+\|_\infty^{-\frac{2p}{2p^*_\alpha-p}} S_{H,L}^{\frac{2p^*_\alpha}{2p^*_\alpha-p}} - Q_0\lambda^{\frac{p}{p-r}} := c_\infty,
$$

which is a contradiction. Thus $\mu_i = 0$ for all i. Hence $J=\phi$ which implies that

$$
\left(\int_\Omega \frac{|u_n(y)|^{p^*_\alpha}}{|x-y|^\alpha}dy\right)|u_n(x)|^{p^*_\alpha} \to \left(\int_\Omega \frac{|u(y)|^{p^*_\alpha}}{|x-y|^\alpha}dy\right)|u(x)|^{p^*_\alpha} \text{ as } n\to\infty.
$$

Moreover, $u_n \to u$ strongly in $L^m(\Omega)$, for $1\leq m<p^*$, then one can calculate that

$$
\left|\int_\Omega f(x)|u_n|^{r-2}u_n(u_n-u)dx\right| \to 0 \text{ as } n\to\infty. \tag{4.5.45}
$$

Since the sequence $u_n \rightharpoonup u$ weakly in $W_0^{2,p}(\Omega)$, therefore sequence $\{u_n-u\}$ is bounded in $W_0^{2,p}(\Omega)$ and $\langle \mathcal{I}'_\lambda(u_n), (u_n-u)\rangle \to 0$ as $n\to\infty$.

$$\langle \mathcal{I}'_\lambda(u_n), (u_n - u)\rangle = \int_\Omega |\Delta u_n|^{p-2}\Delta u_n \Delta(u_n - u)dx - \lambda \int_\Omega f(x)|u_n|^{r-2}u_n(u_n - u)dx$$
$$- \int_\Omega \int_\Omega g(x)g(y)\frac{|u_n(x)|^{p^*_\alpha}|u_n(y)|^{p^*_\alpha - 2}u_n(y)(u_n(y) - u(y))}{|x-y|^\alpha},$$
$$\to 0 \text{ as } n \to \infty.$$

This with Lemma 4.1.1 and (4.5.45) yield that

$$\lim_{n\to\infty}\left(\int_\Omega |\Delta u_n|^{p-2}\Delta u_n \Delta(u_n - u)dx\right) = 0.$$

Then by the same argument used in Lemma 4.4.1, we deduce that $u_n \to u$ strongly in $W^{2,p}_0(\Omega)$. Thus $\mathcal{I}_\lambda$ satisfies the $(PS)_c$-condition for $c \in (-\infty, c_\infty)$, which completes the proof. □

Lemma 4.5.3. *Let* $0 < \lambda < \Upsilon_1$ *and* Υ_1 *be same as in* (4.3.17)*. If* $1 < r < p$ *and* $q = p^*_\alpha$*, then* $\mathcal{I}_\lambda$ *has a minimizer* u_1 *in* $\mathcal{N}^+_\lambda$ *satisfying the following:*

(*i*) $\mathcal{I}_\lambda(u_1) = a_\lambda = a^+_\lambda < 0$*;*

(*ii*) u_1 *is a nontrivial solution;*

(*iii*) $\mathcal{I}_\lambda(u_1) \to 0$ *as* $\lambda \to 0^+$.

Proof. According to the result of By Lemma 4.3.9 (*i*), we say that there exists a $(PS)_{a_\lambda}$-sequence $\{u_n\} \in \mathcal{N}_\lambda$ for $\mathcal{I}_\lambda$ i.e.

$$\lim_{n\to\infty} \mathcal{I}_\lambda(u_n) = a_\lambda \le a^+_\lambda < 0 \quad \text{and} \quad \lim_{n\to\infty} \mathcal{I}'_\lambda(u_n) = 0 \text{ in } X^{-1}.$$

With Q_0 as defined in Lemma 4.5.1, choose $\lambda > 0$ such that

$$c_\infty = \frac{N + 2p - \alpha}{p(2N-\alpha)}\|g^+\|_\infty^{-\frac{2p}{2p^*_\alpha - p}} S_{H,L}^{\frac{2p^*_\alpha}{2p^*_\alpha - p}} - Q_0\lambda^{\frac{p}{p-r}} > 0,$$

for all $0 < \lambda < \Upsilon_1$. Thus $c_\infty > 0 > a_\lambda$. This with Lemma 4.5.2 provides us that $\{u_n\}$ has a convergent subsequence such that $u_n \to u_1$ in $W^{2,p}_0(\Omega)$. Thus by the same idea used in Lemma 4.4.2, we get the required result. □

Proof of Theorem 4.0.3: From Lemma 4.5.3, we conclude that a nontrivial solution $u_1 \in \mathcal{N}_\lambda^+$ exists corresponding to the problem $(\mathcal{G}_\lambda)$ satisfying $\mathcal{I}_\lambda(u_1) \to 0$ as $\lambda \to 0^+$. The proof is complete. □

4.6 Conclusion

In this chapter, we obtain the existence and multiplicity results of the nontrivial solution for p-biharmonic equation with critical and subcritical Choquard nonlinearity involving sign-changing weight functions respectively. To obtain the existence in the critical case, one needs to study the concentration-compactness type Lion's Lemma for Choquard equations in the case of p-biharmonic operator. In our awareness, there is no appearance of such Lemma that elaborates the concentration of a weakly convergent sequence at finite points in the case of the p-biharmonic critical Choquard equation. So our work delivers a contribution to the literature on the p-biharmonic equation with Choquard nonlinearity. In fact, results obtained here are even new for p-Laplcaian. For multiplicity results in the critical case, one needs to study the minimizers which is an open problem even in the case of p-Laplacian.

5

Polyharmonic System With Sign-Changing Nonlinearities

In this chapter, we deal with the existence and multiplicity results of polyharmonic system involving concave-convex type critical nonlinearities with sign-changing weight functions. Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $m \in \mathbb{N}$, $N \geq 2m+1$. Then we consider the following polyharmonic system.

$$(E_{\lambda,\mu}) \begin{cases} (-\Delta)^m u = \lambda f(x)|u|^{r-2}u + \frac{\beta}{\beta+\gamma} h(x)|u|^{\beta-2}u|v|^{\gamma} & \text{in } \Omega, \\ (-\Delta)^m v = \mu g(x)|v|^{r-2}v + \frac{\gamma}{\beta+\gamma} h(x)|u|^{\beta}|v|^{\gamma-2}v & \text{in } \Omega, \\ D^k u = D^k v = 0 \quad \text{for all} \quad |k| \leq m-1 & \text{on } \partial\Omega, \end{cases}$$

where $1 < r < 2$, $\beta > 1$, $\gamma > 1$ satisfying $2 < \beta + \gamma \leq 2^*_m$ with $2^*_m = \frac{2N}{N-2m}$ as a critical Sobolev exponent and λ, μ are the parameter such that $(\lambda, \mu) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$.

Here Δ^m denotes the polyharmonic operator.

The motivation behind the study of polyharmonic system involving critical nonlinearity and sign-changing weight functions is the work of Shang and Li [61]. They investigated the multiplicity results of nontrivial solutions for polyharmonic equation with critical exponents. To construct our problem more precise, we give the following assumptions on the weight functions f, g and h:

($a1$) $f,\ g \in L^{\alpha}(\Omega)$ with $\alpha = \frac{\beta+\gamma}{\beta+\gamma-r}$, $f^{\pm} = \max\{\pm f, 0\} \not\equiv 0$ in $\overline{\Omega}$ and $g^{\pm} = \max\{\pm g, 0\} \not\equiv 0$ in $\overline{\Omega}$ i.e. (f and g are possibly sign-changing on $\overline{\Omega}$).

($h1$) $h \in L^{\infty}(\Omega)$ and $h^{+} = \max\{h, 0\} \not\equiv 0$ in Ω.

5.1 Main results

To state our main results, we introduce

$$\Lambda_1 := \left(\frac{2-r}{(\beta+\gamma-r)\|h\|_{\infty}}\right)^{\frac{2}{\beta+\gamma-2}} \left(\frac{\beta+\gamma-r}{\beta+\gamma-2}\right)^{-\frac{2}{2-r}} S^{\frac{2(\beta+\gamma-r)}{(2-r)(\beta+\gamma-2)}} > 0, \qquad (5.1.1)$$

where S is the best constant which is defined later. Then we obtain the following existence results:

Theorem 5.1.1. *Assume that* ($a1$), ($h1$) *hold. If* $1 \le r < 2 < \frac{N}{m}$, $2 < \beta+\gamma \le 2^*_m$, *and* $\lambda,\ \mu > 0$ *satisfy* $0 < (\lambda\|f\|_{\alpha})^{\frac{2}{2-r}} + (\mu\|g\|_{\alpha})^{\frac{2}{2-r}} < \Lambda_1$, *then the system* $(E_{\lambda,\mu})$ *has at least one nontrivial solution in* $H_0^m(\Omega) \times H_0^m(\Omega)$.

Theorem 5.1.2. *(Second nontrivial solution in subcritical case). Assume that* ($a1$), ($h1$) *hold. If* $1 \le r < 2 < \frac{N}{m}$, $2 < \beta+\gamma < 2^*_m$, *and* $\lambda,\ \mu > 0$ *satisfy* $0 < (\lambda\|f\|_{\alpha})^{\frac{2}{2-r}} + (\mu\|g\|_{\alpha})^{\frac{2}{2-r}} < (\frac{r}{2})^{\frac{2}{2-r}}\Lambda_1$, *then the system* $(E_{\lambda,\mu})$ *has at least two nontrivial solution in* $H_0^m(\Omega) \times H_0^m(\Omega)$.

To obtain the second nontrivial in critical case $\beta+\gamma = 2^*_m$, we need the following extra assumptions on f, g and h:

($a2$) There exist a_0, b_0 and $r_0 > 0$ such that $B(x_0, 2r_0) \subset \Omega$ and $f(x) \ge a_0$, $g(x) \ge b_0$ for all $x \in B(0, 2r_0)$.

(*h*2) There exists $\delta_0 > 0$ such that $\|h\|_\infty = h(0) = \max_{x\in\overline{\Omega}} h(x)$, $h(x) > 0$ for all $x \in B(0, 2r_0)$ and

$$h(x) = h(0) + o\left(|x|^{\delta_0}\right) \text{ as } x \to 0.$$

Then we have the following result:

Theorem 5.1.3. *(Second nontrivial solution in critical case). Assume that* $(a1)-(h2)$ *hold. If* $1 \le r < 2 < \frac{N}{m}$, *and* λ, $\mu > 0$ *satisfy* $0 < (\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} < (\frac{r}{2})^{\frac{2}{2-r}}\Lambda_1$, *then the system* $(E_{\lambda,\mu})$ *has at least two nontrivial solution in* $H_0^m(\Omega) \times H_0^m(\Omega)$.

We firstly define the function space corresponding to the problem $(E_{\lambda,\mu})$, posed in framework of Sobolev space $\mathcal{H} := H_0^m(\Omega) \times H_0^m(\Omega)$ with standard norm

$$\|(u, v)\| = \left(\|D^m u\|^2 + \|D^m v\|^2\right)^{\frac{1}{2}},$$

where

$$\|D^m u\|^2 = \begin{cases} \|(-\Delta)^{\frac{m}{2}} u\|^2 & \text{if } m = 2j, j = 1, 2, \cdots, \\ \|\nabla(-\Delta)^{\frac{m-1}{2}} u\|^2 & \text{if } m = 2j-1, j = 1, 2, \cdots. \end{cases}$$

Then $\mathcal{H}$ is a Hilbert space.

Definition 5.1.1. *A pair of functions* $(u, v) \in \mathcal{H}$ *is said to be a weak solution of system* $(E_{\lambda,\mu})$ *if for all* $(\phi_1, \phi_2) \in \mathcal{H}$, *the following hold,*

(*i*) *when m is even,*

$$\begin{aligned}&\int_\Omega (-\Delta)^{\frac{m}{2}} u(-\Delta)^{\frac{m}{2}}\phi_1 + \int_\Omega (-\Delta)^{\frac{m}{2}} v(-\Delta)^{\frac{m}{2}}\phi_2 - \lambda\int_\Omega f(x)|u|^{r-2}u\phi_1 \\ &- \mu\int_\Omega g(x)|v|^{r-2}v\phi_2 - \frac{\beta}{\beta+\gamma}\int_\Omega h|u|^{\beta-2}u|v|^\gamma\phi_1 - \frac{\gamma}{\beta+\gamma}\int_\Omega h|u|^\beta|v|^{\gamma-2}v\phi_2 = 0.\end{aligned}$$

(*ii*) *when m is odd,*

$$\int_\Omega \nabla(-\Delta)^{\frac{m-1}{2}}u\cdot\nabla(-\Delta)^{\frac{m-1}{2}}\phi_1+\int_\Omega \nabla(-\Delta)^{\frac{m-1}{2}}u\cdot\nabla(-\Delta)^{\frac{m-1}{2}}\phi_2$$
$$-\lambda\int_\Omega f(x)|u|^{r-2}u\phi_1-\mu\int_\Omega g(x)|v|^{r-2}v\phi_2-\frac{\beta}{\beta+\gamma}\int_\Omega h(x)|u|^{\beta-2}u|v|^{\gamma}\phi_1$$
$$-\frac{\gamma}{\beta+\gamma}\int_\Omega h(x)|u|^{\beta}|v|^{\gamma-2}v\phi_2=0.$$

Now, we define the energy functional $I_{\lambda,\mu}:\mathcal{H}\to\mathbb{R}$ associated with the problem $(E_{\lambda,\mu})$ as

$$I_{\lambda,\mu}(u,v)=\frac{1}{2}\|(u,v)\|^2-\frac{1}{r}\int_\Omega(\lambda f(x)|u|^r+\mu g(x)|v|^r)dx-\frac{1}{\beta+\gamma}\int_\Omega h(x)|u|^{\beta}|v|^{\gamma}dx.$$

Then $I_{\lambda,\mu}$ is well defined in $\mathcal{H}$ and $I_{\lambda,\mu}\in C^1(\mathcal{H},\mathbb{R})$. Moreover, the critical points of the functional $I_{\lambda,\mu}$ are the weak solutions of $(E_{\lambda,\mu})$. Further, we will prove a lemma which will be used to prove the second solution in critical case. For this, let S be the best Sobolev constant defined as

$$S:=\inf_{u\in H_0^m(\Omega)\setminus\{0\}}\frac{\|D^m u\|^2}{\left(\int_\Omega |u|^{\beta+\gamma}\right)^{\frac{2}{\beta+\gamma}}},\tag{5.1.2}$$

where $\beta+\gamma=2^*_m$. Then it is well known that S is achieved if and only if $\Omega=\mathbb{R}^N$, by the function $U(x)=\frac{C_{N,m}^{\frac{N-2m}{4m}}}{(1+|x|^2)^{\frac{N-2m}{2}}}$ (see[64]). Moreover, all the minimizers of S are obtained by

$$U_\epsilon(x)=\epsilon^{\frac{2m-N}{2}}U\left(\frac{x}{\epsilon}\right)=\frac{C_{N,m}^{\frac{N-2m}{4m}}\epsilon^{\frac{N-2m}{2}}}{(\epsilon^2+|x|^2)^{\frac{N-2m}{2}}}\quad\text{, where }\epsilon>0.\tag{5.1.3}$$

The normalizing constant $C_{N,m}:=C(N,m)=\prod_{j=1-m}^{m}(N-2j)$ and is chosen in such a way that $U_\epsilon(x)$ solves the equation

$$(-\Delta)^m u=|u|^{2^*_m-2}u\quad\text{ in }\mathbb{R}^N,$$

and satisfies

$$\|U_\epsilon(x)\|^2 = |U_\epsilon(x)|_{2_m^*}^{2_m^*} = S^{\frac{N}{2m}}.$$

Now, consider the following minimization problem

$$S_{\beta,\gamma} = \inf_{(u,v)\in\mathcal{H}\setminus\{(0,0)\}} \frac{\|D^m u\|^2 + \|D^m v\|^2}{\left(\int_\Omega |u|^\beta |v|^\gamma dx\right)^{\frac{2}{\beta+\gamma}}}. \tag{5.1.4}$$

We establish the following relationship between $S_{\beta,\gamma}$ and S, using the idea of [3].

Lemma 5.1.1. *For the constant $S_{\beta,\gamma}$ and S given in* (5.1.2) *and* (5.1.4)*, it holds*

$$S_{\beta,\gamma} = \left[\left(\frac{\beta}{\gamma}\right)^{\frac{\gamma}{\beta+\gamma}} + \left(\frac{\gamma}{\beta}\right)^{\frac{\beta}{\beta+\gamma}}\right] S. \tag{5.1.5}$$

In particular, the constant $S_{\beta,\gamma}$ is achieved for $\Omega = \mathbb{R}^N$.

Proof. Let $\{w_n\} \subset H_0^m(\Omega)$ be a minimizing sequence for S. Then take the sequences $u_n = sw_n$ and $v_n = tw_n$ in $H_0^m(\Omega)$, where s, $t > 0$. By definition of $S_{\beta,\gamma}$, we have

$$S_{\beta,\gamma} \leq \frac{\|(u_n, v_n)\|^2}{\left(\int_\Omega |u_n|^\beta |v_n|^\gamma dx\right)^{\frac{2}{\beta+\gamma}}}.$$

Thus,

$$S_{\beta,\gamma} \leq \frac{(s^2+t^2)S}{s^{\frac{2\beta}{\beta+\gamma}} t^{\frac{2\gamma}{\beta+\gamma}}} = \left[\left(\frac{s}{t}\right)^{\frac{2\gamma}{\beta+\gamma}} + \left(\frac{t}{s}\right)^{\frac{2\beta}{\beta+\gamma}}\right] S.$$

Now, define a function $\Upsilon : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Upsilon(x) = x^{\frac{2\gamma}{\beta+\gamma}} + x^{\frac{-2\beta}{\beta+\gamma}}$. Then $\Upsilon(\frac{s}{t}) = \left(\frac{s}{t}\right)^{\frac{2\gamma}{\beta+\gamma}} + \left(\frac{t}{s}\right)^{\frac{2\beta}{\beta+\gamma}}$ and Υ attains its minimum at $x_0 = \left(\frac{\beta}{\gamma}\right)^{\frac{1}{2}}$. So, we have

$$\min_{x\in\mathbb{R}^+} \Upsilon(x) = \Upsilon(x_0) = \left(\frac{\beta}{\gamma}\right)^{\frac{\gamma}{\beta+\gamma}} + \left(\frac{\gamma}{\beta}\right)^{\frac{\beta}{\beta+\gamma}}.$$

Choosing s, t such that $\frac{s}{t} = \left(\frac{\beta}{\gamma}\right)^{\frac{1}{2}}$ and letting $n \to \infty$ yields

$$S_{\beta,\gamma} \leq \left[\left(\frac{\beta}{\gamma}\right)^{\frac{\gamma}{\beta+\gamma}} + \left(\frac{\gamma}{\beta}\right)^{\frac{\beta}{\beta+\gamma}}\right] S. \tag{5.1.6}$$

On the other hand, let $\{(u_n, v_n)\}$ be a minimizing sequence for $S_{\beta,\gamma}$. Define $a_n = s_n v_n$ for some $s_n > 0$ such that $\int_\Omega |u_n|^{\beta+\gamma} dx = \int_\Omega |a_n|^{\beta+\gamma} dx$. Then Young's inequality implies that

$$\int_\Omega |u_n|^\beta |a_n|^\gamma \leq \frac{\beta}{\beta+\gamma} \int_\Omega |u_n|^{\beta+\gamma} + \frac{\gamma}{\beta+\gamma} \int_\Omega |a_n|^{\beta+\gamma} = \int_\Omega |a_n|^{\beta+\gamma} = \int_\Omega |u_n|^{\beta+\gamma}.$$

Thus, using this we obtain

$$\begin{aligned}
\frac{\|(u_n, v_n)\|^2}{\left(\int_\Omega |u_n|^\beta |v_n|^\gamma dx\right)^{\frac{2}{\beta+\gamma}}} &= s_n^{\frac{2\gamma}{\beta+\gamma}} \left[\frac{\|D^m u_n\|^2}{\left(\int_\Omega |u_n|^\beta |a_n|^\gamma dx\right)^{\frac{2}{\beta+\gamma}}} + \frac{\|D^m v_n\|^2}{\left(\int_\Omega |u_n|^\beta |a_n|^\gamma dx\right)^{\frac{2}{\beta+\gamma}}}\right] \\
&\geq s_n^{\frac{2\gamma}{\beta+\gamma}} \frac{\|D^m u_n\|^2}{\left(\int_\Omega |u_n|^{\beta+\gamma} dx\right)^{\frac{2}{\beta+\gamma}}} + s_n^{\frac{2\gamma}{\beta+\gamma}-2} \frac{\|D^m a_n\|^2}{\left(\int_\Omega |a_n|^{\beta+\gamma} dx\right)^{\frac{2}{\beta+\gamma}}} \\
&\geq \left(s_n^{\frac{2\gamma}{\beta+\gamma}} + s_n^{\frac{2\gamma}{\beta+\gamma}-2}\right) S \geq \Upsilon(x_0) S.
\end{aligned}$$

On passing the limit as $n \to \infty$, we get

$$S_{\beta,\gamma} \geq \left[\left(\frac{\beta}{\gamma}\right)^{\frac{\gamma}{\beta+\gamma}} + \left(\frac{\gamma}{\beta}\right)^{\frac{\beta}{\beta+\gamma}}\right] S. \tag{5.1.7}$$

Hence, from equation (5.1.6) and (5.1.7), we obtain the required result. □

5.2 The Palais-Smale Condition

Lemma 5.2.1. *Suppose that $\{(u_n, v_n)\} \subset \mathcal{H}$ is a $(PS)_c$-sequence for $I_{\lambda,\mu}$ such that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in $\mathcal{H}$. Then $I'_{\lambda,\mu}(u, v) = 0$ and there exists a positive constant P_0 depending on m, N, r and S such that*

$$I_{\lambda,\mu}(u,v) \geq -P_0\left((\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}}\right).$$

Proof. Let $\{(u_n, v_n)\}$ be a $(PS)_c$-sequence in $\mathcal{H}$, then by using the standard argument, one can easily obtain $I'_{\lambda,\mu}(u,v) = 0$, i.e. $\langle I'_{\lambda,\mu}(u,v), (u,v)\rangle = 0$. Using this, Hölder's and Young's inequalities, we obtain

$$\begin{aligned}
I_{\lambda,\mu}(u,v) =& \left(\frac{1}{2} - \frac{1}{\beta+\gamma}\right)\|(u,v)\|^2 - \left(\frac{1}{r} - \frac{1}{\beta+\gamma}\right)\int_\Omega (\lambda f(x)|u|^r + \mu g(x)|v|^r)dx \\
\geq& \frac{m}{N}\|(u,v)\|^2 - \frac{(\beta+\gamma-r)}{r(\beta+\gamma)}S^{-\frac{r}{2}} \\
&\times \left[\omega^{\frac{2}{2-r}}\left(\frac{2-r}{2}\right)\left((\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}}\right) + \frac{\omega^{-\frac{2}{r}}}{2}\|(u,v)\|^2\right] \\
=& \frac{m}{N}\|(u,v)\|^2 - \frac{m}{N}\|(u,v)\|^2 - P_0\left((\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}}\right) \\
=& -P_0\left((\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}}\right),
\end{aligned}$$

where $P_0 = \frac{(\beta+\gamma-r)(2-r)}{2r(\beta+\gamma)}S^{-\frac{r}{2}}\omega^{\frac{2}{2-r}}$, $\omega = \left(\frac{N(\beta+\gamma-r)}{2m(\beta+\gamma)}S^{-\frac{r}{2}}\right)^{\frac{r}{2}}$. This completes the proof. □

Lemma 5.2.2. *If $\{(u_n, v_n)\} \subset \mathcal{H}$ is a $(PS)_c$-sequence for $I_{\lambda,\mu}$, then $\{(u_n, v_n)\}$ is bounded in $\mathcal{H}$.*

Proof. The proof is same as done in Lemma 3.2.2 of chapter 3. □

Lemma 5.2.3. *$I_{\lambda,\mu}$ satisfies the $(PS)_c$-condition with c satisfying $c \in (0, c_\infty)$, where*

$$c_\infty = \frac{m}{N}S_{\beta,\gamma}^{\frac{N}{2m}}\|h\|_\infty^{-\frac{N-2m}{2m}} - P_0\left((\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}}\right),$$

and P_0 is same as given in Lemma 5.2.1.

Proof. Let $\{(u_n, v_n)\} \subset \mathcal{H}$ be a $(PS)_c$-sequence for $I_{\lambda,\mu}$ with $0 < c < c_\infty$. Then by Lemma 5.2.2, $\{(u_n, v_n)\}$ is a bounded sequence in $\mathcal{H}$. Hence, up to a subsequence, $(u_n, v_n) \rightharpoonup (u,v)$ weakly in $\mathcal{H}$. So $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ weakly in $H_0^m(\Omega)$, $u_n \to u$ and $v_n \to v$ strongly in $L^s(\Omega)$ for all $1 \leq s < 2_m^*$ and $u_n \to u$, $v_n \to v$ pointwise a.e.

in Ω. Thus

$$\int_\Omega (\lambda f(x)|u_n|^r + \mu g(x)|v_n|^r)dx = \int_\Omega (\lambda f(x)|u|^r + \mu g(x)|v|^r)dx + o_n(1). \quad (5.2.8)$$

Also, $I'_{\lambda,\mu}(u,v) = 0$, follows from Lemma 5.2.1. Now, define $(\tilde{u}_n, \tilde{v}_n)$, where $\tilde{u}_n = u_n - u$, $\tilde{v}_n = v_n - v$. Then by Brézis-Lieb Lemma [12] and Vitali theorem, we have

$$\|(\tilde{u}_n, \tilde{v}_n)\|^2 = \|(u_n, v_n)\|^2 - \|(u,v)\|^2 + o_n(1),$$
$$\int_\Omega h(x)|\tilde{u}_n|^\beta |\tilde{v}_n|^\gamma dx = \int_\Omega h(x)|u_n|^\beta |v_n|^\gamma dx - \int_\Omega h(x)|u|^\beta |v|^\gamma dx + o_n(1). \quad (5.2.9)$$

Using $I_{\lambda,\mu}(u_n, v_n) = c + o_n(1)$, $I'_{\lambda,\mu}(u_n, v_n) = o_n(1)$, (5.2.8) and (5.2.9), we obtain

$$\frac{1}{2}\|(\tilde{u}_n, \tilde{v}_n)\|^2 - \frac{1}{\beta+\gamma}\int_\Omega h(x)|\tilde{u}_n|^\beta |\tilde{v}_n|^\gamma dx = c - I_{\lambda,\mu}(u,v) + o_n(1), \quad (5.2.10)$$

and

$$\|(\tilde{u}_n, \tilde{v}_n)\|^2 - \int_\Omega h(x)|\tilde{u}_n|^\beta |\tilde{v}_n|^\gamma dx = \langle I'_{\lambda,\mu}(u,v), (u_n - u, v_n - v)\rangle + o_n(1) = o_n(1).$$

Therefore, we assume

$$\|(\tilde{u}_n, \tilde{v}_n)\|^2 \to l, \quad \int_\Omega h(x)|\tilde{u}_n|^\beta |\tilde{v}_n|^\gamma dx \to l. \quad (5.2.11)$$

If $l = 0$, then proof is complete. If $l > 0$, then by definition of $S_{\beta,\gamma}$ and (5.2.11), we get

$$S_{\beta,\gamma} l^{\frac{2}{\beta+\gamma}} \le S_{\beta,\gamma} \lim_{n\to\infty} \left(\|h\|_\infty \int_\Omega |u_n|^\beta |v_n|^\gamma dx\right)^{\frac{2}{2_m^*}} \le \|h\|_\infty^{\frac{2}{\beta+\gamma}} \lim_{n\to\infty} \|(\tilde{u}_n, \tilde{v}_n)\|^2 = \|h\|_\infty^{\frac{2}{\beta+\gamma}} l.$$

As $\beta + \gamma = 2_m^*$, so above relation gives

$$l \ge S_{\beta,\gamma}^{\frac{N}{2m}} \|h\|_\infty^{-\frac{(N-2m)}{2m}}.$$

Now, by (5.2.10), (5.2.11) and Lemma 5.2.1, we obtain

$$c = \left(\frac{1}{2} - \frac{1}{\beta+\gamma}\right) l + I_{\lambda,\mu}(u,v) \geq \frac{m}{N} S_{\beta,\gamma}^{\frac{N}{2m}} \|h\|_\infty^{-\frac{N-2m}{2m}} - P_0 \left((\lambda \|f\|_\alpha)^{\frac{2}{2-r}} + (\mu \|g\|_\alpha)^{\frac{2}{2-r}} \right) = c_\infty,$$

which is a contradiction to $c < c_\infty$. Hence, proof is completed. □

5.3 Nehari Manifold for $(E_{\lambda,\mu})$

Since the energy functional $I_{\lambda,\mu}$ is not bounded below on $\mathcal{H}$, it is appropriate to consider the functional on the Nehari manifold

$$\mathcal{N}_{\lambda,\mu} = \left\{ (u,v) \in \mathcal{H} \setminus \{(0,0)\} : \langle I'_{\lambda,\mu}(u,v), (u,v) \rangle = 0 \right\}.$$

Thus, $(u,v) \in \mathcal{N}_{\lambda,\mu}$ if and only if

$$\langle I'_{\lambda,\mu}(u,v), (u,v) \rangle = \|(u,v)\|^2 - Q_{\lambda,\mu}(u,v) - \int_\Omega h(x)|u|^\beta |v|^\gamma dx = 0. \quad (5.3.12)$$

It is easy to see that $\mathcal{N}_{\lambda,\mu}$ contains every nonzero solution of the system $(E_{\lambda,\mu})$. In fact, we will show later that local minimizers of $\mathcal{N}_{\lambda,\mu}$ are the critical points of $I_{\lambda,\mu}$.

Lemma 5.3.1. *The energy functional $I_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$.*

Proof. Let $(u,v) \in \mathcal{N}_{\lambda,\mu}$, then by (5.3.12), Hölder inequality and Sobolev embedding theorem, we have

$$\begin{aligned} I_{\lambda,\mu}(u,v) =& \frac{\beta+\gamma-2}{2(\beta+\gamma)} \|(u,v)\|^2 - \frac{\beta+\gamma-r}{r(\beta+\gamma)} Q_{\lambda,\mu}(u,v) \\ \geq& \frac{\beta+\gamma-2}{2(\beta+\gamma)} \|(u,v)\|^2 - \frac{\beta+\gamma-r}{r(\beta+\gamma)} S^{-\frac{r}{2}} \left((\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} \right)^{\frac{2-r}{2}} \|(u,v)\|^r. \end{aligned} \quad (5.3.13)$$

Since $1 < r < 2$. Thus, $I_{\lambda,\mu}$ is coercive.

Now, consider the function $\rho : \mathbb{R} \to \mathbb{R}$ as $\rho(t) = at^2 - bt^r$. Then one can easily see that $\rho'(t) = 0$ if and only if $t = \left(\frac{br}{2a}\right)^{\frac{1}{2-r}} := t^*$ and $\rho''(t^*) > 0$. So ρ attains its minimum at t^*. Moreover,

$$\rho(t) \geq \rho(t^*) = -(2-r)\left(\frac{b}{2}\right)^{\frac{2}{2-r}}\left(\frac{r}{a}\right)^{\frac{r}{2-r}}.$$

Taking $a = \frac{\beta+\gamma-2}{2(\beta+\gamma)}$, $b = \frac{\beta+\gamma-r}{r(\beta+\gamma)}S^{-\frac{r}{2}}\left((\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}}$ and $t = \|(u,v)\|$ in the function ρ, we obtain

$$I_{\lambda,\mu}(u,v) \geq \rho(\|(u,v)\|) \geq \rho(t^*).$$

Hence, $I_{\lambda,\mu}$ is bounded below on $\mathcal{N}_{\lambda,\mu}$. $\square$

The Nehari manifold is closely related to the fibering map introduced by Drábek and Pohozaev in [24]. For each (u,v), we define $\Psi_{(u,v)} : t \to I_{\lambda,\mu}(tu,tv)$ given by

$$\begin{aligned}
\Psi_{(u,v)}(t) =& I_{\lambda,\mu}(tu,tv) = \frac{t^2}{2}\|(u,v)\|^2 - \frac{t^r}{r}Q_{\lambda,\mu}(u,v) - \frac{t^{\beta+\gamma}}{\beta+\gamma}\int_\Omega h(x)|u|^\beta|v|^\gamma dx, \\
\Psi'_{(u,v)}(t) =& t\|(u,v)\|^2 - t^{r-1}Q_{\lambda,\mu}(u,v) - t^{\beta+\gamma-1}\int_\Omega h(x)|u|^\beta|v|^\gamma dx, \\
\Psi''_{(u,v)}(t) =& \|(u,v)\|^2 - (r-1)t^{r-2}Q_{\lambda,\mu}(u,v) - (\beta+\gamma-1)t^{\beta+\gamma-2}\int_\Omega h(x)|u|^\beta|v|^\gamma dx.
\end{aligned}$$

It is observed that $\Psi'_{(u,v)}(t) = 0$ if and only if $(tu,tv) \in \mathcal{N}_{\lambda,\mu}$. Thus $(u,v) \in \mathcal{N}_{\lambda,\mu}$ if and only if $\Psi'_{(u,v)}(1) = 0$. Therefore it is natural to split $\mathcal{N}_{\lambda,\mu}$ into three parts corresponding to local minima, local maxima and point of inflexion respectively as:

$$\begin{aligned}
\mathcal{N}^{\pm}_{\lambda,\mu} &:= \left\{(u,v) \in \mathcal{N}_{\lambda,\mu} : \Psi''_{(u,v)}(1) \gtrless 0\right\}, \\
\mathcal{N}^{0}_{\lambda,\mu} &:= \left\{(u,v) \in \mathcal{N}_{\lambda,\mu} : \Psi''_{(u,v)}(1) = 0\right\}.
\end{aligned}$$

For every $(u,v) \in \mathcal{N}_{\lambda,\mu}$, we have

$$\Psi''_{(u,v)}(1) = \begin{cases} 2\|(u,v)\|^2 - rQ_{\lambda,\mu}(u,v) - (\beta+\gamma)\int_\Omega h(x)|u|^\beta|v|^\gamma dx \\ (2-r)\|(u,v)\|^2 - (\beta+\gamma-r)\int_\Omega h(x)|u|^\beta|v|^\gamma dx \\ (\beta+\gamma-r)Q_{\lambda,\mu}(u,v) - (\beta+\gamma-2)\|(u,v)\|^2. \end{cases} \tag{5.3.14}$$

Now, we have the following Lemma.

Lemma 5.3.2. *If (u_0, v_0) is the local minimizer for $I_{\lambda,\mu}$ on $\mathcal{N}_\lambda$ and $(u_0, v_0) \notin \mathcal{N}^0_{\lambda,\mu}$. Then $I'_{\lambda,\mu}((u_0, v_0)) = 0$ in $\mathcal{H}^{-1}$, where $\mathcal{H}^{-1}$ denotes the dual space of $\mathcal{H}$.*

Proof. The proof follows the same as done in Lemma 1.3.1 in chapter 1. □

Lemma 5.3.3. *We have the following*
(i) If $(u, v) \in \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu}$, then $Q_{\lambda,\mu}(u, v) > 0$.
(ii) If $(u, v) \in \mathcal{N}^-_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu}$, then $\int_\Omega h(x)|u|^\beta|v|^\gamma dx > 0$.

Proof. The proof is directly followed by (5.3.14). □

Now, we will show that $\mathcal{N}^+_{\lambda,\mu}$ and $\mathcal{N}^-_{\lambda,\mu}$ are nonempty. For this we define some notations. For each $(u, v) \in \mathcal{H}$ with $\int_\Omega h(x)|u|^\beta|v|^\gamma dx > 0$

$$t_{\max} = \left(\frac{(2-r)\|(u, v)\|^2}{(\beta+\gamma-r)\int_\Omega h(x)|u|^\beta|v|^\gamma dx} \right)^{\frac{1}{\beta+\gamma-2}} > 0,$$

and for $Q_{\lambda,\mu}(u, v) > 0$

$$\bar{t}_{\max} = \left(\frac{(\beta+\gamma-r)Q_{\lambda,\mu}(u, v)}{(\beta+\gamma-2)\|(u, v)\|^2} \right)^{\frac{1}{2-r}} > 0.$$

Lemma 5.3.4. *Suppose that $0 < (\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} < \Lambda_1$ and $(u, v) \in \mathcal{H}$ then, we have the following:*

(i) If $\int_\Omega h(x)|u|^\beta|v|^\gamma dx > 0$ and $Q_{\lambda,\mu}(u, v) \le 0$, then there exists a unique $t^- > t_{\max}$ such that $(t^-u, t^-v) \in \mathcal{N}^-_{\lambda,\mu}$ and $I_{\lambda,\mu}(t^-u, t^-v) = \sup_{t \ge t_{\max}} I_{\lambda,\mu}(tu, tv)$.

(ii) If $\int_\Omega h(x)|u|^\beta|v|^\gamma dx > 0$ and $Q_{\lambda,\mu}(u, v) > 0$, then there exists a unique $0 < t^+ < t_{\max} < t^-$ such that $(t^+u, t^+v) \in \mathcal{N}^+_{\lambda,\mu}$, $(t^-u, t^-v) \in \mathcal{N}^-_{\lambda,\mu}$. Moreover,

$$I_{\lambda,\mu}(t^+u, t^+v) = \inf_{0 \le t \le t_{max}} I_{\lambda,\mu}(tu, tv); \qquad I_{\lambda,\mu}(t^-u, t^-v) = \sup_{t \ge t_{\max}} I_{\lambda,\mu}(tu, tv).$$

(iii) If $Q_{\lambda,\mu}(u, v) > 0$ and $\int_\Omega h(x)|u|^\beta|v|^\gamma \le 0$, then there exists a unique $0 < t^+ < \bar{t}_{\max}$ such that $(t^+u, t^+v) \in \mathcal{N}^+_{\lambda,\mu}$ and $I_{\lambda,\mu}(t^+u, t^+v) = \inf_{t \ge 0} I_{\lambda,\mu}(tu, tv)$.

(iv) *If* $Q_{\lambda,\mu}(u,v) < 0$ *and* $\int_\Omega h(x)|u|^\beta|v|^\gamma dx < 0$, *then there does not exist any critical point.*

Proof. For $(u,v) \in \mathcal{H}$ with $\int_\Omega h(x)|u|^\beta|v|^\gamma dx > 0$. Define

$$\xi_{(u,v)}(t) = t^{2-r}\|(u,v)\|^2 - t^{\beta+\gamma-r}\int_\Omega h(x)|u|^\beta|v|^\gamma dx, \quad \text{for} \quad t > 0.$$

We have $\xi_{(u,v)}(0) = 0$, $\xi_{(u,v)}(t) \to -\infty$ as $t \to \infty$.
Since

$$\xi'_{(u,v)}(t) = (2-r)t^{1-r}\|(u,v)\|^2 - (\beta+\gamma-r)t^{\beta+\gamma-r-1}\int_\Omega h(x)|u|^\beta|v|^\gamma dx,$$

we get $\xi'_{(u,v)}(t) = 0$ at $t = t_{\max}$, $\xi'_{(u,v)}(t) > 0$ for $t \in [0, t_{\max})$ and $\xi'_{(u,v)}(t) < 0$ for $t \in (t_{\max}, \infty)$. So $\xi_{(u,v)}(t)$ attains its maximum at $t_{\max}$. $\xi_{(u,v)}(t)$ is increasing function for $t \in [0, t_{\max})$ and decreasing for $t \in (t_{\max}, \infty)$. Moreover,

$$\begin{aligned}
\xi_{(u,v)}(t_{\max}) = &\left(\frac{(2-r)\|(u,v)\|^2}{(\beta+\gamma-r)\int_\Omega h(x)|u|^\beta|v|^\gamma dx}\right)^{\frac{2-r}{\beta+\gamma-2}} \|(u,v)\|^2 \\
&- \left(\frac{(2-r)\|(u,v)\|^2}{(\beta+\gamma-r)\int_\Omega h(x)|u|^\beta|v|^\gamma dx}\right)^{\frac{\beta+\gamma-r}{\beta+\gamma-2}} \int_\Omega h(x)|u|^\beta|v|^\gamma dx \\
=&\|(u,v)\|^r \left(\frac{2-r}{\beta+\gamma-r}\right)^{\frac{2-r}{\beta+\gamma-2}} \left(\frac{\beta+\gamma-2}{\beta+\gamma-r}\right) \left(\frac{\|(u,v)\|^{\beta+\gamma}}{\int_\Omega h(x)|u|^\beta|v|^\gamma dx}\right)^{\frac{2-r}{\beta+\gamma-2}} \\
\geq&\|(u,v)\|^r \left(\frac{2-r}{\beta+\gamma-r}\right)^{\frac{2-r}{\beta+\gamma-2}} \left(\frac{\beta+\gamma-2}{\beta+\gamma-r}\right) \left(\frac{S^{\frac{\beta+\gamma}{2}}}{\|h\|_\infty}\right)^{\frac{2-r}{\beta+\gamma-2}}.
\end{aligned}$$

(i) If $\int_\Omega h(x)|u|^\beta|v|^\gamma dx > 0$ and $Q_{\lambda,\mu}(u,v) \leq 0$, there is a unique $t^- > t_{\max} > 0$ such that $\xi_{(u,v)}(t^-) = Q_{\lambda,\mu}(u,v) \leq 0$ and $\xi'_{(u,v)}(t^-) < 0$.

$$\begin{aligned}
\langle I'_{\lambda,\mu}(t^-u, t^-v), (t^-u, t^-v)\rangle =&(t^-)^2\|(u,v)\|^2 - (t^-)^r Q_{\lambda,\mu}(u,v) \\
&- (t^-)^{\beta+\gamma}\int_\Omega h|u|^\beta|v|^\gamma \\
=&(t^-)^r\left(\xi_{(u,v)}(t^-) - Q_{\lambda,\mu}(u,v)\right) = 0.
\end{aligned}$$

Therefore, $(t^- u, t^- v) \in \mathcal{N}_{\lambda,\mu}$.

$$\begin{aligned}\Psi''_{(u,v)}(t^-) &= (2-r)(t^-)^2 \|(u,v)\|^2 - (\beta+\gamma-r)(t^-)^{\beta+\gamma} \int_\Omega h(x)|u|^\beta |v|^\gamma dx \\ &= (t^-)^{1+r} \xi'_{(u,v)}(t^-) < 0.\end{aligned}$$

Hence, $(t^- u, t^- v) \in \mathcal{N}^-_{\lambda,\mu}$. Since for $t > t_{\max}$, we have

$$\Psi''_{(u,v)}(t) = (2-r)t^2\|(u,v)\|^2 - (\beta+\gamma-r)t^{\beta+\gamma} \int_\Omega h|u|^\beta |v|^\gamma = t^{1+r}\xi'_{(u,v)}(t) < 0.$$

$$\frac{d^2}{dt^2} I_{\lambda,\mu}(tu,tv) = (r-1)t^{r-2}\left(\xi_{(u,v)}(t) - Q_{\lambda,\mu}(u,v)\right) + t^{r-1}\xi'_{(u,v)}(t) < 0, \text{ for } t = t^-$$

$$\frac{d}{dt} I_{\lambda,\mu}(tu,tv) = t^{r-1}\left(\xi_{(u,v)}(t) - Q_{\lambda,\mu}(u,v)\right) = 0, \text{ for } t = t^-.$$

Thus,

$$I_{\lambda,\mu}(t^- u, t^- v) = \sup_{t \geq t_{\max}} I_{\lambda,\mu}(tu,tv).$$

(ii) If $\int_\Omega h(x)|u|^\beta |v|^\gamma dx > 0$ and $Q_{\lambda,\mu}(u,v) > 0$, then by (5.3.17)

$$\begin{aligned}\xi_{(u,v)}(0) = 0 < Q_{\lambda,\mu}(u,v) &\leq S^{-\frac{r}{2}} \left((\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}} \|(u,v)\|^r \\ &< \left(\frac{2-r}{\beta+\gamma-r}\right)^{\frac{2-r}{\beta+\gamma-2}} \left(\frac{\beta+\gamma-2}{\beta+\gamma-r}\right) \left(\frac{S^{\frac{\beta+\gamma}{2}}}{\|h\|_\infty}\right)^{\frac{2-r}{\beta+\gamma-2}} \|(u,v)\|^r \\ &\leq \xi_{(u,v)}(t_{\max}),\end{aligned}$$

for $0 < (\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} < \Lambda_1$. There are unique t^+ and t^- such that $0 < t^+ < t_{\max} < t^-$ with

$$\xi_{(u,v)}(t^+) = Q_{\lambda,\mu}(u,v) = \xi_{(u,v)}(t^-) \text{ and } \xi'_{(u,v)}(t^+) > 0 > \xi'_{(u,v)}(t^-).$$

This implies $(t^+ u, t^+ v) \in \mathcal{N}^+_{\lambda,\mu}$, $(t^- u, t^- v) \in \mathcal{N}^-_{\lambda,\mu}$ and

$$\frac{d}{dt}I_{\lambda,\mu}(tu,tv)=0,\ \text{when } t=t^{+} \text{ and } t=t^{-},$$
$$\frac{d^2}{dt^2}I_{\lambda,\mu}(tu,tv)>0,\ \text{when } t\in(0,t_{\max}),$$
$$\frac{d^2}{dt^2}I_{\lambda,\mu}(tu,tv)<0,\ \text{when } t\in(t_{\max},\infty).$$

Thus, we have

$$I_{\lambda,\mu}(t^{+}u,t^{+}v)=\inf_{0\le t\le t_{\max}} I_{\lambda,\mu}(tu,tv),\ I_{\lambda,\mu}(t^{-}u,t^{-}v)=\sup_{t\ge t_{\max}} I_{\lambda,\mu}(tu,tv).$$

(iii) For $(u,v)\in\mathcal{H}$ with $Q_{\lambda,\mu}(u,v)>0$ and $\int_\Omega h(x)|u|^\beta|v|^\gamma dx\le 0$, define

$$\overline{\xi}_{(u,v)}(t)=t^{2-\beta-\gamma}\|(u,v)\|^2-t^{r-\beta-\gamma}Q_{\lambda,\mu}(u,v),\ \text{for } t>0.$$

We have $\overline{\xi}_{(u,v)}(t)\to-\infty$ as $t\to0$, $\overline{\xi}_{(u,v)}(t)\to0$ as $t\to\infty$.
Since

$$\overline{\xi}'_{(u,v)}(t)=(2-\beta-\gamma)t^{1-\beta-\gamma}\|(u,v)\|^2-(r-\beta-\gamma)t^{r-\beta-\gamma-1}Q_{\lambda,\mu}(u,v),$$

we get $\overline{\xi}'_{(u,v)}(t)=0$ at $t=\overline{t}_{\max}$, $\overline{\xi}'_{(u,v)}(t)>0$ for $t\in(0,\overline{t}_{\max})$ and $\overline{\xi}'_{(u,v)}(t)<0$ for $t\in(\overline{t}_{\max},\infty)$. So $\overline{\xi}_{(u,v)}(t)$ attains its maximum at $\overline{t}_{\max}$. $\overline{\xi}_{(u,v)}(t)$ is increasing function for $t\in(0,\overline{t}_{\max})$ and decreasing for $t\in(\overline{t}_{\max},\infty)$. Now, using the same argument used in previous parts, there exists a unique $0<t^{+}<\overline{t}_{\max}$ such that $\overline{\xi}_{(u,v)}(t^{+})=\int_\Omega h(x)|u|^\beta|v|^\gamma dx\le0$, $\overline{\xi}'_{(u,v)}(t^{+})>0$. Also, $\langle I'_{\lambda,\mu}(t^{+}u,t^{+}v),(t^{+}u,t^{+}v)\rangle=0$. Thus, $(t^{+}u,t^{+}v)\in\mathcal{N}_{\lambda,\mu}$. Further $\Psi''_{(u,v)}(t^{+})>0$ so $(t^{+}u,t^{+}v)\in\mathcal{N}^{+}_{\lambda,\mu}$. Since $0<t^{+}<\overline{t}_{\max}$, then $\Psi''_{(u,v)}(t)>0$. Moreover, for $t=t^{+}$, $\frac{d^2}{dt^2}I_{\lambda,\mu}(tu,tv)>0$ and $\frac{d}{dt}I_{\lambda,\mu}(tu,tv)=0$. Hence,

$$I_{\lambda,\mu}(t^{+}u,t^{+}v)=\inf_{t\ge0}I_{\lambda,\mu}(tu,tv).$$

(iv) If $Q_{\lambda,\mu}(u,v)<0$ and $\int_\Omega h(x)|u|^\beta|v|^\gamma dx<0$, then $\Psi_{(u,v)}(0)=0$, $\Psi'_{(u,v)}(t)>0$

for all $t > 0$. This implies $\Psi_{(u,v)}$ is strictly increasing function and does not have critical point.

This completes the proof of Lemma. $\square$

Lemma 5.3.5. *If* $0 < (\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} < \Lambda_1$, *then* $\mathcal{N}^0_{\lambda,\mu} = \emptyset$.

Proof. On contrary, assume that there exists $(\lambda,\mu) \in \mathbb{R}^2 \setminus \{(0,0)\}$ with $0 < (\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} < \Lambda_1$, such that $\mathcal{N}^0_{\lambda,\mu} \neq \emptyset$. Then for $(u,v) \in \mathcal{N}^0_{\lambda,\mu}$, using (5.3.14), we get

$$\|(u,v)\|^2 = \frac{\beta+\gamma-r}{2-r}\int_\Omega h(x)|u|^\beta|v|^\gamma dx \text{ and } \|(u,v)\|^2 = \frac{\beta+\gamma-r}{\beta+\gamma-2}Q_{\lambda,\mu}(u,v). \tag{5.3.15}$$

Now, by Young's inequality and Sobolev embedding theorem, we have

$$\begin{aligned}\int_\Omega h(x)|u|^\beta|v|^\gamma dx \leq& \|h\|_\infty\left(\frac{\beta}{\beta+\gamma}\int_\Omega |u|^{\beta+\gamma}dx + \frac{\gamma}{\beta+\gamma}\int_\Omega |v|^{\beta+\gamma}dx\right)\\ \leq& \|h\|_\infty S^{-\frac{\beta+\gamma}{2}}\|(u,v)\|^{\beta+\gamma}. \end{aligned} \tag{5.3.16}$$

Similarly, by Hölder's inequality and Sobolev embedding theorem, we get

$$Q_{\lambda,\mu}(u,v) \leq S^{-\frac{r}{2}}\left((\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}}\|(u,v)\|^r. \tag{5.3.17}$$

Thus, by (5.3.15), (5.3.16) and (5.3.17), we obtain

$$\|(u,v)\| \geq \left(\frac{2-r}{\beta+\gamma-r}\frac{S^{\frac{\beta+\gamma}{2}}}{\|h\|_\infty}\right)^{\frac{1}{\beta+\gamma-2}} \tag{5.3.18}$$

and

$$\|(u,v)\| \leq \left(\frac{\beta+\gamma-r}{\beta+\gamma-2}\right)^{\frac{1}{2-r}} S^{-\frac{r}{2(2-r)}}\left((\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}}\right)^{\frac{1}{2}}. \tag{5.3.19}$$

On combining (5.3.18) and (5.3.19), we have

$$(\lambda\|f\|_\alpha)^{\frac{2}{2-r}}+(\mu\|g\|_\alpha)^{\frac{2}{2-r}} \geq \left(\frac{2-r}{(\beta+\gamma-r)\|h\|_\infty}\right)^{\frac{2}{\beta+\gamma-2}} \left(\frac{\beta+\gamma-r}{\beta+\gamma-2}\right)^{-\frac{2}{2-r}} S^{\frac{2(\beta+\gamma-r)}{(2-r)(\beta+\gamma-2)}} := \Lambda_1,$$

which is a contradiction. This completes the proof. □

Remark: Thus, from Lemma 5.3.5, if $0 < (\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} < \Lambda_1$, then $\mathcal{N}_{\lambda,\mu} = \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu}$.

Define

$$\theta_{\lambda,\mu} = \inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}} I_{\lambda,\mu}(u,v);\ \theta^\pm_{\lambda,\mu} = \inf_{(u,v)\in\mathcal{N}^\pm_{\lambda,\mu}} I_{\lambda,\mu}(u,v).$$

Now, we end this section with the following result.

Theorem 5.3.1. *The following facts hold:*

(*i*) *If* $0 < (\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} < \Lambda_1$, *then* $\theta_{\lambda,\mu} \leq \theta^+_{\lambda,\mu} < 0$.

(*ii*) *If* $0 < (\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} < \left(\frac{r}{2}\right)^{\frac{2}{2-r}} \Lambda_1$, *then* $\theta^-_{\lambda,\mu} > d_0$, *where* d_0 *is a positive constant depending on* λ, μ, r, N, S, $\|f\|_\alpha$, $\|g\|_\alpha$ *and* $\|h\|_\infty$.

Proof. (*i*) Assume $(u,v) \in \mathcal{N}^+_{\lambda,\mu}$. Then by (5.3.14), we have

$$\frac{2-r}{\beta+\gamma-r}\|(u,v)\|^2 > \int_\Omega h(x)|u|^\beta|v|^\gamma dx. \tag{5.3.20}$$

Using (5.3.12) and (5.3.20), we obtain

$$\begin{aligned} I_{\lambda,\mu}(u,v) &= \left(\frac{1}{2}-\frac{1}{r}\right)\|(u,v)\|^2 + \left(\frac{1}{r}-\frac{1}{\beta+\gamma}\right)\int_\Omega h(x)|u|^\beta|v|^\gamma dx \\ &< \left[\left(\frac{1}{2}-\frac{1}{r}\right) + \left(\frac{1}{r}-\frac{1}{\beta+\gamma}\right)\frac{2-r}{\beta+\gamma-r}\right]\|(u,v)\|^2 \\ &= -\frac{(2-r)(\beta+\gamma-2)}{2r(\beta+\gamma)}\|(u,v)\|^2 < 0. \end{aligned}$$

So, from the definitions of $\theta_{\lambda,\mu}$, $\theta^+_{\lambda,\mu}$, we can deduce that $\theta_{\lambda,\mu} \leq \theta^+_{\lambda,\mu} < 0$.

(*ii*) Let $(u,v) \in \mathcal{N}^-_{\lambda,\mu}$. Then from (5.3.14),

$$\frac{2-r}{\beta+\gamma-r}\|(u,v)\|^2 < \int_\Omega h(x)|u|^\beta|v|^\gamma dx. \tag{5.3.21}$$

Hölder's inequality and Sobolev embedding theorem imply that

$$\|(u,v)\| > \left(\frac{2-r}{(\beta+\gamma-r)\|h\|_\infty}\right)^{\frac{1}{\beta+\gamma-2}} S^{\frac{\beta+\gamma}{2(\beta+\gamma-2)}} \text{ for all } (u,v)\in\mathcal{N}^-_{\lambda,\mu}. \tag{5.3.22}$$

By (5.3.13) and (5.3.22), it follows that

$$\begin{aligned} I_{\lambda,\mu}(u,v) \geq& \|(u,v)\|^r \left[\frac{\beta+\gamma-2}{2(\beta+\gamma)}\|(u,v)\|^{2-r} - \frac{\beta+\gamma-r}{r(\beta+\gamma)S^{\frac{r}{2}}}\left((\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}}\right] \\ >& \left(\frac{2-r}{\beta+\gamma-r}\right)^{\frac{r}{\beta+\gamma-2}} S^{\frac{r(\beta+\gamma)}{2(\beta+\gamma-2)}} \left[\frac{\beta+\gamma-2}{2(\beta+\gamma)}\left(\frac{2-r}{(\beta+\gamma-r)\|h\|_\infty}\right)^{\frac{2-r}{\beta+\gamma-2}} S^{\frac{(2-r)(\beta+\gamma)}{2(\beta+\gamma-2)}} \right. \\ & \left. -\frac{\beta+\gamma-r}{r(\beta+\gamma)}S^{-\frac{r}{2}}\left((\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}}\right]. \end{aligned}$$

Thus, if $0 < (\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} < \left(\frac{r}{2}\right)^{\frac{2}{2-r}}\Lambda_1$, then $I_{\lambda,\mu}(u,v) > d_0$ for all $(u,v)\in \mathcal{N}^-_{\lambda,\mu}$, for some positive constant $d_0 = d_0(\lambda,\mu,r,N,S,\|f\|_\alpha,\|g\|_\alpha,\|h\|_\infty)$. □

5.4 Existence of first solution

In this section, we show the existence of Palais-Smale sequence in $\mathcal{N}^{\pm}_\lambda$ and give the proof of Theorem 5.1.1 and Theorem 5.1.2 respectively.

Lemma 5.4.1. *Suppose* $0 < (\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} < \Lambda_1$, *where* Λ_1 *is same as given in* (5.1.1). *Then for every* $z = (u,v) \in \mathcal{N}_{\lambda,\mu}$, *there exist* $\epsilon > 0$ *and a differentiable mapping* $\zeta : B(0,\epsilon) \subset \mathcal{H} \to \mathbb{R}^+$ *such that* $\zeta(0) = 1$, $\zeta(w)(z-w) \in \mathcal{N}_{\lambda,\mu}$ *and for all* $w = (w_1, w_2) \in \mathcal{H}$

$$\langle \zeta'(0), w\rangle = \frac{2\mathcal{B}(z,w) - r\mathcal{Q}_{\lambda,\mu}(z,w) - 2\mathcal{P}(z,w)}{(2-r)\|(u,v)\|^2 - (\beta+\gamma-r)\int_\Omega h(x)|u|^\beta|v|^\gamma dx}, \tag{5.4.23}$$

where

$$\begin{aligned} \mathcal{B}(z,w) =& \int_\Omega D^m u\cdot D^m w_1 dx + \int_\Omega D^m v\cdot D^m w_2 dx, \\ \mathcal{Q}_{\lambda,\mu}(z,w) =& \lambda\int_\Omega f(x)|u|^{r-2}uw_1 dx + \mu\int_\Omega g(x)|v|^{r-2}vw_2 dx, \end{aligned}$$

$$\mathcal{P}(z,w)=\int_\Omega \beta|u|^{\beta-2}|v|^\gamma u w_1 dx+\int_\Omega \gamma|u|^\beta|v|^{\gamma-2}v w_2 dx.$$

Proof. For $z=(u,v)\in\mathcal{N}_{\lambda,\mu}$, define a map $\vartheta_z:\mathbb{R}\times\mathcal{H}\to\mathbb{R}$ such that

$$\begin{aligned}\vartheta_z(\zeta,w)=\langle I'_{\lambda,\mu}(\zeta(z-w)),\zeta(z-w)\rangle=\zeta^2\|(u-w_1,v-w_2)\|^2\\ -\zeta^r\int_\Omega(\lambda f|u-w_1|^r+\mu g|v-w_2|^r)-\zeta^{\beta+\gamma}\int_\Omega h|u-w_1|^\beta|v-w_2|^\gamma\end{aligned}$$

Then $\vartheta_z(1,(0,0))=\langle I'_{\lambda,\mu}(z),z\rangle=0$ and

$$\begin{aligned}\frac{d}{d\zeta}\vartheta_z(1,(0,0))=&2\|(u,v)\|^2-r\int_\Omega(\lambda f|u|^r+\mu g|v|^r)-(\beta+\gamma)\int_\Omega h|u|^\beta|v|^\gamma\\ =&(2-r)\|(u,v)\|^2-(\beta+\gamma-r)\int_\Omega h|u|^\beta|v|^\gamma\neq 0.\end{aligned}$$

Now, by Implicit Function Theorem, $\exists\ \epsilon>0$ and a differentiable mapping $\zeta: B(0,\epsilon)\subset\mathcal{H}\to\mathbb{R}^+$ such that $\zeta(0)=1$,

$$\langle\zeta'(0),w\rangle=\frac{2\mathcal{B}(z,w)-r\mathcal{Q}_{\lambda,\mu}(z,w)-2\mathcal{P}(z,w)}{(2-r)\|(u,v)\|^2-(\beta+\gamma-r)\int_\Omega h(x)|u|^\beta|v|^\gamma dx},$$

$\vartheta_z(\zeta(w),w)=0\ \forall\ w\in B(0,\epsilon)$. Thus,

$$\langle I'_{\lambda,\mu}(\zeta(w)(z-w)),\zeta(w)(z-w)\rangle=0\ \forall\ w\in B(0,\epsilon).$$

Therefore, $\zeta(w)(z-w)\in\mathcal{N}_{\lambda,\mu}$. □

Lemma 5.4.2. *Suppose* $0<(\lambda\|f\|_\alpha)^{\frac{2}{2-r}}+(\mu\|g\|_\alpha)^{\frac{2}{2-r}}<\Lambda_1$, *where* Λ_1 *is same as given in* (5.1.1). *Then for every* $z=(u,v)\in\mathcal{N}^-_{\lambda,\mu}$, *there exist* $\epsilon>0$ *and a differentiable map* $\zeta^-:B(0,\epsilon)\subset\mathcal{H}\to\mathbb{R}^+$ *such that* $\zeta^-(0)=1$ *and* $\zeta^-(w)(z-w)\in\mathcal{N}^-_{\lambda,\mu}$. *Moreover, for all* $(w_1,w_2)\in\mathcal{H}$

$$\langle(\zeta^-)'(0),w\rangle=\frac{2\mathcal{B}(z,w)-r\mathcal{Q}_{\lambda,\mu}(z,w)-2\mathcal{P}(z,w)}{(2-r)\|(u,v)\|^2-(\beta+\gamma-r)\int_\Omega h(x)|u|^\beta|v|^\gamma dx},$$

where $\mathcal{B}$, $\mathcal{Q}_{\lambda,\mu}$ and $\mathcal{P}$ are defined same as in Lemma 5.4.1.

Proof. By the same argument used in Lemma 5.4.1, we get the required result. $\square$

Lemma 5.4.3. *Let $1 \leq r < 2 < \frac{N}{m}$ and $2 < \beta + \gamma \leq 2_m^*$, then the following results hold:*

(*i*) *If $0 < (\lambda \|f\|_\alpha)^{\frac{2}{2-r}} + (\mu \|g\|_\alpha)^{\frac{2}{2-r}} < \Lambda_1$, then there exists a $(PS)_{\theta_{\lambda,\mu}}$-sequence $\{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\mu}$ in $\mathcal{H}$ for $I_{\lambda,\mu}$.*

(*ii*) *If $0 < (\lambda \|f\|_\alpha)^{\frac{2}{2-r}} + (\mu \|g\|_\alpha)^{\frac{2}{2-r}} < \left(\frac{r}{2}\right)^{\frac{2}{2-r}} \Lambda_1$, then there exists a $(PS)_{\theta^-_{\lambda,\mu}}$-sequence $\{(u_n, v_n)\} \subset \mathcal{N}^-_{\lambda,\mu}$ in $\mathcal{H}$ for $I_{\lambda,\mu}$, where Λ_1 is same as given in* (5.1.1).

Proof. (*i*) By Lemma 5.3.1 and Ekeland Variational Principle [27], there exists a minimizing sequence $\{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\mu}$ such that

$$I_{\lambda,\mu}(u_n, v_n) < \theta_{\lambda,\mu} + \frac{1}{n},$$
$$I_{\lambda,\mu}(u_n, v_n) < I_{\lambda,\mu}(u, v) + \frac{1}{n}\|(u, v) - (u_n, v_n)\|, \text{ for each } (u, v) \in \mathcal{N}_{\lambda,\mu}.$$

Since $\theta_{\lambda,\mu} < 0$ and taking n large, we get

$$\begin{aligned} I_{\lambda,\mu}(u_n, v_n) = &\left(\frac{1}{2} - \frac{1}{\beta+\gamma}\right)\|(u_n, v_n)\|^2 - \left(\frac{1}{r} - \frac{1}{\beta+\gamma}\right)\int_\Omega (\lambda f |u_n|^r + \mu g |v_n|^r) \\ <&\theta_{\lambda,\mu} + \frac{1}{n} < \frac{\theta_{\lambda,\mu}}{2}. \end{aligned} \tag{5.4.24}$$

Thus, we have

$$\begin{aligned} 0 < -\frac{r(\beta+\gamma)\theta_{\lambda,\mu}}{2(\beta+\gamma-r)} < &\int_\Omega (\lambda f(x)|u_n|^r + \mu g(x)|v_n|^r)\, dx \\ \leq& S^{-\frac{r}{2}} \left((\lambda \|f\|_\alpha)^{\frac{2}{2-r}} + (\mu \|g\|_\alpha)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}} \|(u_n, v_n)\|^r. \end{aligned} \tag{5.4.25}$$

Consequently, $(u_n, v_n) \neq (0, 0)$. Also, (5.4.24), (5.4.25) and Hölder's inequality assert that

$$\|(u_n, v_n)\| \leq \left[\frac{2(\beta+\gamma-r)}{r(\beta+\gamma-2)} S^{-\frac{r}{2}} \left((\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}}\right]^{\frac{1}{2-r}},$$

and

$$\|(u_n, v_n)\| \geq \left[-\frac{r(\beta+\gamma)}{2(\beta+\gamma-r)} \theta_{\lambda,\mu} S^{\frac{r}{2}} \left((\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}}\right)^{\frac{r-2}{2}}\right]^{\frac{1}{r}}.$$

Now, we will show that

$$\|I'_{\lambda,\mu}(u_n, v_n)\|_{\mathcal{H}^{-1}} \to 0, \text{ as } n \to \infty.$$

for polyharmonic system by following the same idea as used in Lemma 2.3.9 of chapter 2.

Now, we will show that $\|\zeta_n'(0)\|$ is uniformly bounded. From (5.4.23), by Hölder's inequality and Sobolev embedding, we have

$$|\langle \zeta_n'(0)\rangle| \leq \frac{M_1\|(w_1, w_2)\|}{|(2-r)\|(u_n, v_n)\|^2 - (\beta+\gamma-r)\int_\Omega h(x)|u_n|^\beta |v_n|^\gamma dx|},$$

for some $M_1 > 0$.

We are left to show that

$$|(2-r)\|(u_n, v_n)\|^2 - (\beta+\gamma-r)\int_\Omega h(x)|u_n|^\beta |v_n|^\gamma dx| \geq M_2,$$

for some $M_2 > 0$ and n is taking large enough. On contrary, suppose there exists a subsequence $\{(u_n, v_n)\}$ such that

$$(2-r)\|(u_n, v_n)\|^2 - (\beta+\gamma-r)\int_\Omega h(x)|u_n|^\beta |v_n|^\gamma dx = o_n(1). \tag{5.4.26}$$

From (5.4.26) and using $(u_n, v_n) \in \mathcal{N}_\lambda$, we have

$$\begin{aligned}\|(u_n, v_n)\|^2 =& \frac{\beta+\gamma-r}{2-r}\int_\Omega h(x)|u_n|^\beta |v_n|^\gamma dx + o_n(1),\\ \|(u_n, v_n)\|^2 =& \frac{\beta+\gamma-r}{\beta+\gamma-2} Q_{\lambda,\mu}(u_n, v_n) + o_n(1).\end{aligned}$$

By Hölder's inequality and Sobolev embedding theorem, we obtain

$$\begin{aligned}\|(u_n, v_n)\| &\geq \left(\frac{2-r}{\beta+\gamma-r}\frac{S^{\frac{\beta+\gamma}{2}}}{\|h\|_\infty}\right)^{\frac{1}{\beta+\gamma-2}} + o_n(1),\\ \|(u_n, v_n)\| &\leq \left(\frac{\beta+\gamma-r}{\beta+\gamma-2}\right)^{\frac{1}{2-r}} S^{-\frac{r}{2(2-r)}} \left((\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}}\right)^{\frac{1}{2}} + o_n(1).\end{aligned}$$

This implies that $(\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} \geq \Lambda_1$, which is a contradiction to the fact that $0 < (\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} < \Lambda_1$. Hence,

$$\left\langle I'_{\lambda,\mu}(u_n, v_n), \frac{(u, v)}{\|(u, v)\|}\right\rangle \leq \frac{M}{n}.$$

Thus, proof of (i) is completed.

(ii) By Lemma 5.4.2, proof of (ii) can be shown in similar way as above. □

Now, we will show the existence of a local minimum for $I_{\lambda,\mu}$ on $\mathcal{N}^+_{\lambda,\mu}$.

Theorem 5.4.1. *Let Λ_1 be the same defined as in* (5.1.1). *If $1 \leq r < 2 < \frac{N}{m}$, $2 < \beta+\gamma \leq 2^*_m$, and $0 < (\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} < \Lambda_1$, then $I_{\lambda,\mu}$ has a minimizer $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ in $\mathcal{N}^+_{\lambda,\mu}$ and it satisfies the following:*

(i) $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = \theta_{\lambda,\mu} = \theta^+_{\lambda,\mu} < 0$.

(ii) $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ *is a nontrivial solution of the system* $(E_{\lambda,\mu})$.

(iii) $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \to (0,0)$ *as* $\lambda \to 0^+$, $\mu \to 0^+$.

Proof. By Lemma 5.4.3 (i), there exists a minimizing sequence $\{(u_n, v_n)\}$ for $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$ such that

$$I_{\lambda,\mu}(u_n, v_n) = \theta_{\lambda,\mu} + o_n(1),\ I'_{\lambda,\mu}(u_n, v_n) = o_n(1) \text{ in } \mathcal{H}^{-1}. \tag{5.4.27}$$

By coercivity of $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$, we obtain that $\{(u_n, v_n)\}$ is bounded in $\mathcal{H}$. Therefore up to a subsequence still denoted by $\{(u_n, v_n)\}$ converges weakly to $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{H}$. This implies

$$\begin{aligned} &u_n \rightharpoonup u^1_{\lambda,\mu},\ v_n \rightharpoonup v^1_{\lambda,\mu} \text{ weakly in } H^m_0(\Omega),\\ &u_n \rightharpoonup u^1_{\lambda,\mu},\ v_n \rightharpoonup v^1_{\lambda,\mu} \text{ a.e. } \Omega,\\ &u_n \rightharpoonup u^1_{\lambda,\mu},\ v_n \rightharpoonup v^1_{\lambda,\mu} \text{ strongly in } L^s(\Omega)\ \forall\ 1 \le s < 2^*_m. \end{aligned} \tag{5.4.28}$$

It is easy to see that as $n \to \infty$

$$Q_{\lambda,\mu}(u_n, v_n) = Q_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) + o_n(1). \tag{5.4.29}$$

First we claim that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is a nontrivial solution of system $(E_{\lambda,\mu})$. From (5.4.27) and (5.4.28), one can easily verify that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is a weak solution of the system $(E_{\lambda,\mu})$. Since $(u_n, v_n) \in \mathcal{N}_{\lambda,\mu}$ and by the definition of $I_{\lambda,\mu}$, we have

$$Q_{\lambda,\mu}(u_n, v_n) = \frac{r(\beta+\gamma-2)}{2(\beta+\gamma-r)}\|(u_n, v_n)\|^2 - \frac{r(\beta+\gamma)}{(\beta+\gamma-r)} I_{\lambda,\mu}(u_n, v_n). \tag{5.4.30}$$

Then letting $n \to \infty$ in (5.4.30) and using (5.4.27), (5.4.29) with $\theta_{\lambda,\mu} < 0$, we obtain

$$Q_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \ge -\frac{r(\beta+\gamma)}{(\beta+\gamma-r)}\theta_{\lambda,\mu} > 0.$$

Thus, $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}_{\lambda,\mu}$ is a nontrivial solution of the system $(E_{\lambda,\mu})$.
Now, we show that $(u_n, v_n) \to (u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ strongly in $\mathcal{H}$ and $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = \theta_{\lambda,\mu}$. If $(u, v) \in \mathcal{N}_{\lambda,\mu}$, then

$$I_{\lambda,\mu}(u, v) = \frac{\beta+\gamma-2}{2(\beta+\gamma)}\|(u, v)\|^2 - \frac{\beta+\gamma-r}{r(\beta+\gamma)} Q_{\lambda,\mu}(u, v). \tag{5.4.31}$$

In order to prove that $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = \theta_{\lambda,\mu}$, it is sufficient to recall that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}_{\lambda,\mu}$, (5.4.31) and apply Fatou's lemma to obtain

$$
\begin{aligned}
\theta_{\lambda,\mu} \leq I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) &= \frac{\beta+\gamma-2}{2(\beta+\gamma)}\|(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})\|^2 - \frac{\beta+\gamma-r}{r(\beta+\gamma)}Q_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \\
&\leq \liminf_{n\to\infty}\left(\frac{\beta+\gamma-2}{2(\beta+\gamma)}\|(u_n, v_n)\|^2 - \frac{(\beta+\gamma-r)}{r(\beta+\gamma)}Q_{\lambda,\mu}(u_n, v_n)\right) \qquad (5.4.32) \\
&\leq \liminf_{n\to\infty} I_{\lambda,\mu}(u_n, v_n) = \theta_{\lambda,\mu}.
\end{aligned}
$$

This implies that $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = \theta_{\lambda,\mu}$ and $\lim\limits_{n\to\infty}\|(u_n, v_n)\|^2 = \|(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})\|^2$. Let $(\overline{u}_n, \overline{v}_n) = (u_n - u^1_{\lambda,\mu}, v_n - v^1_{\lambda,\mu})$, then by Brézis and Lieb lemma [12] gives

$$
\|(\overline{u}_n, \overline{v}_n)\|^2 = \|(u_n, v_n)\|^2 - \|(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})\|^2 + o_n(1).
$$

Therefore, $(u_n, v_n) \to (u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ strongly in $\mathcal{H}$. Moreover, we have $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$. Thus, $\theta_{\lambda,\mu} = \theta^+_{\lambda,\mu}$. On contrary, if $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda}$, then using (5.3.21) and (5.4.32), we have that $\int_\Omega h(x)|u^1_{\lambda,\mu}|^\beta|v^1_{\lambda,\mu}|^\gamma > 0$ and $Q_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) > 0$. Thus, from Lemma 5.3.4, there exist unique t^+_1 and t^-_1 such that $(t^+_1 u^1_{\lambda,\mu}, t^+_1 v^1_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$ and $(t^-_1 u^1_{\lambda,\mu}, t^-_1 v^1_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$. In particular, we have $t^+_1 < t^-_1 = 1$. Since

$$
\frac{d}{dt}I_{\lambda,\mu}(t^+_1 u^1_{\lambda,\mu}, t^+_1 v^1_{\lambda,\mu}) = 0,\ \frac{d^2}{dt^2}I_{\lambda,\mu}(t^+_1 u^1_{\lambda,\mu}, t^+_1 v^1_{\lambda,\mu}) > 0,
$$

there exists $t^+_1 < \bar{t} \leq t^-_1$ such that $I_{\lambda,\mu}(t^+_1 u^1_{\lambda,\mu}, t^+_1 v^1_{\lambda,\mu}) < I_{\lambda,\mu}(\bar{t}u^1_{\lambda,\mu}, \bar{t}v^1_{\lambda,\mu})$. By Lemma 5.3.4, we obtain

$$
I_{\lambda,\mu}(t^+_1 u^1_{\lambda,\mu}, t^+_1 v^1_{\lambda,\mu}) < I_{\lambda,\mu}(\bar{t}u^1_{\lambda,\mu}, \bar{t}v^1_{\lambda,\mu}) \leq I_{\lambda,\mu}(t^-_1 u^1_{\lambda,\mu}, t^-_1 v^1_{\lambda,\mu}) = I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = \theta_{\lambda,\mu},
$$

which is a contradiction. Therefore, using Lemma 5.3.2, we can conclude that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is a nontrivial solution of the system $(E_{\lambda,\mu})$.

(iii) Further, from Theorem 5.3.1 (i) and (5.3.13), we have

$$0 > \theta^+_{\lambda,\mu} \geq \theta_{\lambda,\mu} = I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) > -\frac{\beta+\gamma-r}{r(\beta+\gamma)} S^{\frac{-r}{2}} \left((\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} \right)^{\frac{2-r}{2}} \times \|(u,v)\|^r.$$

Which implies $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \to (0,0)$ as $(\lambda,\mu) \to (0^+,0^+)$, $\mu \to 0^+$ and completes the proof. □

Theorem 5.4.2. *If* $1 \leq r < 2 < \frac{N}{m}$, $2 < \beta+\gamma < 2^*_m$ *and* $0 < (\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} < \left(\frac{r}{2}\right)^{\frac{2}{2-r}} \Lambda_1$, *then* $I_{\lambda,\mu}$ *has a minimizer* $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ *in* $\mathcal{N}^-_\lambda$ *and satisfies the following:*

(i) $I_{\lambda,\mu}(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) = \theta^-_{\lambda,\mu}$.

(ii) $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ *is a solution of the system* $(E_{\lambda,\mu})$.

Proof. Let $\{(u_n,v_n)\}$ be a minimizing sequence for $I_{\lambda,\mu}$ on $\mathcal{N}^-_\lambda$. Then by $I_{\lambda,\mu}$ coercive on $\mathcal{N}_\lambda$ and the compact imbedding theorem, there exist a subsequence $\{(u_n,v_n)\}$ and $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) \in \mathcal{H}$ such that $u_n \rightharpoonup u^2_{\lambda,\mu}$, $v_n \rightharpoonup v^2_{\lambda,\mu}$ weakly in $H^m_0(\Omega)$, $u_n \to u^2_{\lambda,\mu}$, $v_n \to v^2_{\lambda,\mu}$ strongly in $L^r(\Omega)$, $L^{\beta+\gamma}(\Omega)$. This implies

$$Q_{\lambda,\mu}(u_n,v_n) = Q_{\lambda,\mu}(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) + o_n(1), \int_\Omega h(x)|u_n|^\beta|v_n|^\gamma = \int_\Omega h(x)|u^2_{\lambda,\mu}|^\beta|v^2_{\lambda,\mu}|^\gamma + o_n(1).$$

Using (5.3.21) and (5.3.22), there exists $M_3 > 0$ such that $\int_\Omega h(x)|u_n|^\beta|v_n|^\gamma dx > M_3$. This implies that

$$\int_\Omega h(x)|u^2_{\lambda,\mu}|^\beta|v^2_{\lambda,\mu}|^\gamma dx \geq M_3.$$

Now, we prove that $(u_n,v_n) \to (u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ strongly in $\mathcal{H}$. On contrary, we assume that $\|(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})\| < \liminf_{n\to\infty} \|(u_n,v_n)\|$. Then using Lemma 5.3.4, there exists a unique t^-_2 such that $(t^-_2 u^2_{\lambda,\mu}, t^-_2 v^2_{\lambda,\mu}) \in \mathcal{N}^-_\lambda$. Since $(u_n,v_n) \in \mathcal{N}^-_\lambda$, $I_{\lambda,\mu}(u_n,v_n) \geq I_{\lambda,\mu}(tu_n,tv_n)$ for all $t \geq 0$, we have

$$\theta^-_{\lambda,\mu} \leq I_{\lambda,\mu}(t^- u^2_{\lambda,\mu}, t^- v^2_{\lambda,\mu}) < \lim_{n\to\infty} I_{\lambda,\mu}(t^- u_n, t^- v_n) \leq \lim_{n\to\infty} I_{\lambda,\mu}(u_n,v_n) = \theta^-_{\lambda,\mu}.$$

Hence, $(u_n, v_n) \to (u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ strongly in $\mathcal{H}$. This implies

$$I_{\lambda,\mu}(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) = \lim_{n\to\infty} I_{\lambda,\mu}(u_n, v_n) = \theta^-_{\lambda,\mu}.$$

By Lemma 5.3.2 and (5.4.32), we say that $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ is a nontrivial solution of the system $(E_{\lambda,\mu})$. Finally, by using the same arguments as in the proof of Theorem 5.4.1, for all $0 < (\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} < \left(\frac{r}{2}\right)^{\frac{2}{2-r}} \Lambda_1$, we have that $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ is a solution of the system $(E_{\lambda,\mu})$. □

Proof of Theorems 5.1.1 and 5.1.2: Theorem 5.1.1 and Theorem 5.1.2 follow from Theorem 5.4.1 and from Theorem 5.4.2 respectively. Also from Theorem 5.4.1 and 5.4.2, we obtain that for all $1 < r < 2 < \frac{N}{m}$, $2 < \beta + \gamma < 2^*_m$, λ, $\mu > 0$ and $0 < (\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} < \left(\frac{r}{2}\right)^{\frac{2}{2-r}} \Lambda_1$, the system $(E_{\lambda,\mu})$ has two nontrivial solutions $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^+_\lambda$ and $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) \in \mathcal{N}^-_\lambda$. Since $\mathcal{N}^+_\lambda \cap \mathcal{N}^-_\lambda = \phi$, we can conclude that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ and $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ are distinct. This completes the proof.

5.5 Existence of a second solution

In this section, we will show the existence of a second weak solution in the critical case $\beta+\gamma = 2^*_m$ as a limit of Palais-Smale sequence which is obtained as minimizing sequence for $I_{\lambda,\mu}$ in $\mathcal{N}^-_\lambda$.

For this, taking $\rho > 0$ small enough such that $B(0,\rho) \subset \Omega$ and define the function $u_\epsilon(x) = \phi(x)U_\epsilon(x)$, where $\phi(x) \in C^\infty_0(B(0,\rho))$ is a cut-off function such that $\phi(x) \equiv 1$ in $B(0,\rho)$ and $U_\epsilon(x)$ is same as mentioned in (5.1.3). Then, we have the following estimates (see [35, 40, 63]).

Lemma 5.5.1. *Suppose $N \geq 2m + 1$. Then the following estimates hold when $\epsilon \to 0$:*

$$\|u_\epsilon\|^2 = S^{\frac{N}{2m}} + O(\epsilon^{N-2m}). \tag{5.5.33}$$

$$\int_\Omega |u_\epsilon|^{2^*_m} dx = S^{\frac{N}{2m}} + O(\epsilon^N). \tag{5.5.34}$$

$$\int_\Omega |u_\epsilon|^r dx = \begin{cases} O_1(\epsilon^{\frac{(N-2m)r}{2}}) & \text{if } 1 < r < \frac{N}{N-2m}, \\ O_1(\epsilon^{N-\frac{(N-2m)r}{2}} |\ln \epsilon|) & \text{if } r = \frac{N}{N-2m}, \\ O_1(\epsilon^{N-\frac{(N-2m)r}{2}}) & \text{if } \frac{N}{N-2m} < r < 2^*_m. \end{cases} \tag{5.5.35}$$

Lemma 5.5.2. *Suppose that (a1), (a2), (h1), (h2) hold with* $\delta_0 > N - 2m$ *and* $\frac{N}{N-2m} \leq r < 2$. *Then there exists* $\overline{\Lambda} > 0$ *such that for all* $0 < (\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} < \overline{\Lambda}$ *there exists* $(u_{\lambda,\mu}, v_{\lambda,\mu})$ *in* $\mathcal{H} \setminus \{(0,0)\}$ *such that*

$$\sup_{t\geq 0} I_{\lambda,\mu}(tu_{\lambda,\mu}, tv_{\lambda,\mu}) < c_\infty,$$

where c_∞ *is the constant given in Lemma 5.2.3.*
In particular, $\theta^-_{\lambda,\mu} < c_\infty$ *for all* $0 < (\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}} < \overline{\Lambda}$.

Proof. By assumption $(h2)$, there exists $\delta_0 > N - 2m$ such that, for $x \in B(0, 2\rho_0)$ where $0 < \rho_0 \leq r_0$

$$h(x) = h(0) + o\left(|x|^{\delta_0}\right) \text{ as } x \to 0.$$

Define a functional $\tau : \mathcal{H} \to \mathbb{R}$ such that

$$\tau(u,v) = \frac{1}{2}\|(u,v)\|^2 - \frac{1}{\beta+\gamma}\int_\Omega h(x)|u|^\beta |v|^\gamma dx \;\forall\; (u,v) \in \mathcal{H}. \tag{5.5.36}$$

Set $\overline{u}_\epsilon = \sqrt{\beta} u_\epsilon$, $\overline{v}_\epsilon = \sqrt{\gamma} u_\epsilon$ with $(\overline{u}_\epsilon, \overline{v}_\epsilon) \in \mathcal{H}$. The map $\tau(t\overline{u}_\epsilon, t\overline{v}_\epsilon)$ satisfies $\tau(0) = 0$, $\tau(t\overline{u}_\epsilon, t\overline{v}_\epsilon) > 0$ for $t > 0$ small and $\tau(t\overline{u}_\epsilon, t\overline{v}_\epsilon) < 0$ for $t > 0$ large. Moreover, τ attains its maximum at

$$t_0 = \left(\frac{\|(\overline{u}_\epsilon, \overline{v}_\epsilon)\|^2}{\int_\Omega h(x)|\overline{u}_\epsilon|^\beta |\overline{v}_\epsilon|^\gamma dx}\right)^{\frac{1}{\beta+\gamma-2}}. \tag{5.5.37}$$

Thus, from (5.5.33), (5.5.34), (5.5.36), (5.5.37) and (5.1.5), we have

$$\begin{aligned}
\sup_{t\geq 0}\tau(t\overline{u}_\epsilon, t\overline{v}_\epsilon) =&\frac{t_0^2}{2}\|(\overline{u}_\epsilon,\overline{v}_\epsilon)\|^2 - \frac{t_0^{\beta+\gamma}}{\beta+\gamma}\int_\Omega h(x)|\overline{u}_\epsilon|^\beta|\overline{v}_\epsilon|^\gamma dx\\
=&\left(\frac{1}{2}-\frac{1}{\beta+\gamma}\right)\frac{\|(\overline{u}_\epsilon,\overline{v}_\epsilon)\|^{\frac{2(\beta+\gamma)}{\beta+\gamma-2}}}{\left(\int_\Omega h(x)|\overline{u}_\epsilon|^\beta|\overline{v}_\epsilon|^\gamma dx\right)^{\frac{2}{\beta+\gamma-2}}}\\
=&\frac{m}{N}\left[\left(\frac{\beta}{\gamma}\right)^{\frac{\gamma}{\beta+\gamma}}+\left(\frac{\gamma}{\beta}\right)^{\frac{\beta}{\beta+\gamma}}\right]^{\frac{N}{2m}}\left[\frac{\|u_\epsilon\|^2}{\left(\int_\Omega h(x)|u_\epsilon|^{2_m^*}dx\right)^{\frac{2}{2_m^*}}}\right]^{\frac{N}{2m}}\\
=&\frac{m}{N}\left[\left(\frac{\beta}{\gamma}\right)^{\frac{\gamma}{\beta+\gamma}}+\left(\frac{\gamma}{\beta}\right)^{\frac{\beta}{\beta+\gamma}}\right]^{\frac{N}{2m}}\left[\frac{S^{\frac{N}{2m}}+O(\epsilon^{N-2m})}{\left(h(0)S^{\frac{N}{2m}}+O(\epsilon^N)+O(\epsilon^{\delta_0})\right)^{\frac{2}{2_m^*}}}\right]^{\frac{N}{2m}}\\
\leq&\frac{m}{N}(h(0))^{-\frac{N-2m}{2m}}S_{\beta,\gamma}^{\frac{N}{2m}}+O(\epsilon^{N-2m})-O(\epsilon^{\delta_0}).
\end{aligned}$$

Therefore,

$$\sup_{t\geq 0}\tau(t\overline{u}_\epsilon, t\overline{v}_\epsilon) \leq\frac{m}{N}(h(0))^{-\frac{N-2m}{2m}}S_{\beta,\gamma}^{\frac{N}{2m}}+O(\epsilon^{N-2m})-O(\epsilon^{\delta_0}). \tag{5.5.38}$$

Now, we choose $\delta_1>0$ such that $c_\infty>0$ for all $0<(\lambda\|f\|_\alpha)^{\frac{2}{2-r}}+(\mu\|g\|_\alpha)^{\frac{2}{2-r}}<\delta_1$. Using the definition of $I_{\lambda,\mu}$ and λ, $\mu>0$, we obtain $I_{\lambda,\mu}(t\overline{u}_\epsilon, t\overline{v}_\epsilon)\leq\frac{t^2}{2}\|(\overline{u}_\epsilon,\overline{v}_\epsilon)\|^2$ for $t\geq 0$. Thus, there exists $t_0\in(0,1)$ such that

$$\sup_{0\leq t\leq t_0} I_{\lambda,\mu}(t\overline{u}_\epsilon, t\overline{v}_\epsilon)<c_\infty\ \forall\ 0<(\lambda\|f\|_\alpha)^{\frac{2}{2-r}}+(\mu\|g\|_\alpha)^{\frac{2}{2-r}}<\delta_1.$$

Using $\beta,\gamma>1$, (5.5.38) and (5.5.35), we obtain

$$\begin{aligned}
\sup_{t\geq t_0} I_{\lambda,\mu}(t\overline{u}_\epsilon, t\overline{v}_\epsilon)&=\sup_{t\geq t_0}\left(\tau(t\overline{u}_\epsilon, t\overline{v}_\epsilon)-\frac{1}{r}Q_{\lambda,\mu}(t\overline{u}_\epsilon, t\overline{v}_\epsilon)\right)\\
&\leq\frac{m}{N}(h(0))^{-\frac{N-2m}{2m}}S_{\beta,\gamma}^{\frac{N}{2m}}+O(\epsilon^{N-2m})-\frac{t_0^r}{r}\int_\Omega(\lambda f|\overline{u}_\epsilon|^r+\mu g|\overline{v}_\epsilon|^r)\\
&\leq\frac{m}{N}(h(0))^{-\frac{N-2m}{2m}}S_{\beta,\gamma}^{\frac{N}{2m}}+O(\epsilon^{N-2m})-\frac{t_0^r}{r}\left(a_0\lambda\beta^{\frac{r}{2}}+b_0\mu\gamma^{\frac{r}{2}}\right)\int_\Omega|u_\epsilon|^r dx
\end{aligned}$$

$$\leq \frac{m}{N}(h(0))^{-\frac{N-2m}{2m}} S_{\beta,\gamma}^{\frac{N}{2m}} + O(\epsilon^{N-2m}) - \frac{t_0^r}{r}\eta\,(\lambda+\mu)\int_\Omega |u_\epsilon|^r dx$$

$$\leq \frac{m}{N}(h(0))^{-\frac{N-2m}{2m}} S_{\beta,\gamma}^{\frac{N}{2m}} + O(\epsilon^{N-2m})$$

$$- \frac{1}{r}t_0^r\eta(\lambda+\mu)\begin{cases} O_1\left(\epsilon^{N-\frac{(N-2m)r}{2}}|\ln\epsilon|\right) \text{ if } r=\frac{N}{N-2m} \\ O_1\left(\epsilon^{N-\frac{(N-2m)r}{2}}\right) \text{ if } \frac{N}{N-2m} < r < 2_m^*, \end{cases}$$

where $\eta = \min\{a_0, b_0\}$. Choose $\delta_2 > 0$ in such a way that $0 \leq \epsilon < \delta_2$. Now, take $\epsilon = \left((\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}}\right)^{\frac{1}{N-2m}}$. Then, we have

$$\sup_{t\geq t_0} I_{\lambda,\mu}(t\overline{u}_\epsilon, t\overline{v}_\epsilon) \leq \frac{m}{N}(h(0))^{-\frac{N-2m}{2m}} S_{\beta,\gamma}^{\frac{N}{2m}} + O(\mathcal{A}(\lambda,\mu))$$

$$- \frac{\eta(\lambda+\mu)}{r}\begin{cases} O_1\left((\mathcal{A}(\lambda,\mu))^{\frac{N}{2(N-2m)}}\,|\ln(\mathcal{A}(\lambda,\mu))|\right) \text{ if } r=\frac{N}{N-2m} \\ O_1\left((\mathcal{A}(\lambda,\mu))^{\frac{N}{N-2m}-\frac{r}{2}}\right) \text{ if } \frac{N}{N-2m} < r < 2_m^*, \end{cases} \tag{5.5.39}$$

where $\mathcal{A}(\lambda,\mu) = (\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}}$.

Case(i): When $r = \frac{N}{N-2m}$, we can choose $\delta_3 > 0$ with $0 < \mathcal{A}(\lambda,\mu) < \delta_3$ such that

$$O(\mathcal{A}(\lambda,\mu)) - \frac{\eta(\lambda+\mu)}{r}O_1\left((\mathcal{A}(\lambda,\mu))^{\frac{N}{2(N-2m)}}\,|\ln(\mathcal{A}(\lambda,\mu))|\right) < -P_0\left(\mathcal{A}(\lambda,\mu)\right),$$

as $\lambda,\mu \to 0,\ |\ln(\mathcal{A}(\lambda,\mu))| \to +\infty$.

Case(ii): When $\frac{N}{N-2m} < r < 2_m^*$, we can choose $\delta_4 > 0$ with $0 < \mathcal{A}(\lambda,\mu) < \delta_4$ such that

$$O(\mathcal{A}(\lambda,\mu)) - \frac{\eta(\lambda+\mu)}{r}O_1\left((\mathcal{A}(\lambda,\mu))^{\frac{N}{N-2m}-\frac{r}{2}}\right) < -P_0\left(\mathcal{A}(\lambda,\mu)\right),$$

as $1 + \frac{2}{2-r}\left(\frac{N}{N-2m} - \frac{r}{2}\right) < \frac{2}{2-r}$ if and only if $r > \frac{N}{N-2m}$.

Now, choose $\overline{\Lambda} = \min\{\delta_1, \delta_2^{N-2m}, \delta_3, \delta_4\} > 0$. Then using this together with (5.5.39), we have

$$\sup_{t\geq 0} I_{\lambda,\mu}(t\overline{u}_\epsilon, t\overline{v}_\epsilon) < \frac{m}{N}(h(0))^{-\frac{N-2m}{2m}} S_{\beta,\gamma}^{\frac{N}{2m}} - P_0\left((\lambda\|f\|_\alpha)^{\frac{2}{2-r}} + (\mu\|g\|_\alpha)^{\frac{2}{2-r}}\right) = c_\infty, \tag{5.5.40}$$

for $0<(\lambda\|f\|_\alpha)^{\frac{2}{2-r}}+(\mu\|g\|_\alpha)^{\frac{2}{2-r}}<\overline{\Lambda}$.

Next, we show that $\theta^-_{\lambda,\mu}<c_\infty$ for all $0<(\lambda\|f\|_\alpha)^{\frac{2}{2-r}}+(\mu\|g\|_\alpha)^{\frac{2}{2-r}}<\overline{\Lambda}$. From $(a2)$, $(h2)$ and the definition of $(\overline{u}_\epsilon,\overline{v}_\epsilon)$, we get

$$\int_\Omega h(x)|\overline{u}_\epsilon|^\beta|\overline{v}_\epsilon|^\gamma dx>0,\quad Q_{\lambda,\mu}(\overline{u}_\epsilon,\overline{v}_\epsilon)>0.$$

Combining this with Lemma 5.3.4 (ii), definition of $\theta^-_{\lambda,\mu}$ and (5.5.40), for all $0<(\lambda\|f\|_\alpha)^{\frac{2}{2-r}}+(\mu\|g\|_\alpha)^{\frac{2}{2-r}}<\overline{\Lambda}$, we obtain that there exists $t_{\lambda,\mu}>0$ such that $(t_{\lambda,\mu}\overline{u}_\epsilon,t_{\lambda,\mu}\overline{v}_\epsilon)\in\mathcal{N}^-_\lambda$ with

$$\theta^-_{\lambda,\mu}\le I_{\lambda,\mu}(t_{\lambda,\mu}\overline{u}_\epsilon,t_{\lambda,\mu}\overline{v}_\epsilon)<\sup_{t\ge0}I_{\lambda,\mu}(t\overline{u}_\epsilon,t\overline{v}_\epsilon)<c_\infty.$$

On taking $(\overline{u}_\epsilon,\overline{v}_\epsilon)=(u_{\lambda,\mu},v_{\lambda,\mu})$, we obtain the desirable result which completes the proof. □

Theorem 5.5.1. *Assume that* $(a1)$, $(a2)$, $(h1)$ *and* $(h2)$ *hold. Then* $I_{\lambda,\mu}$ *satisfies the* $(PS)_{\theta^-_{\lambda,\mu}}$ *condition for all* $0<(\lambda\|f\|_\alpha)^{\frac{2}{2-r}}+(\mu\|g\|_\alpha)^{\frac{2}{2-r}}<\left(\frac{r}{2}\right)^{\frac{2}{2-r}}\Lambda_1$*. Moreover,* $I_{\lambda,\mu}$ *has a minimizer* $(u^2_{\lambda,\mu},v^2_{\lambda,\mu})$ *in* $\mathcal{N}^-_\lambda$ *and satisfies the following conditions:*

- (i) $I_{\lambda,\mu}(u^2_{\lambda,\mu},v^2_{\lambda,\mu})=\theta^-_{\lambda,\mu}>0$.
- (ii) $(u^2_{\lambda,\mu},v^2_{\lambda,\mu})$ *is a nontrivial solution of the system* $(E_{\lambda,\mu})$,

  where Λ_1 *is same as mentioned in* (5.1.1).

Proof. By virtue of Lemma 5.4.3 (ii), for $0<(\lambda\|f\|_\alpha)^{\frac{2}{2-r}}+(\mu\|g\|_\alpha)^{\frac{2}{2-r}}<\left(\frac{r}{2}\right)^{\frac{2}{2-r}}\Lambda_1$, there exists a $(PS)_{\theta^-_{\lambda,\mu}}$-sequence $\{(u_n,v_n)\}\subset\mathcal{N}^-_\lambda$ in $\mathcal{H}$ for $I_{\lambda,\mu}$. Then, from Lemma 5.2.2, we find that $\{(u_n,v_n)\}$ is bounded in $\mathcal{H}$. Now, using Lemma 5.5.2 and Lemma 5.2.3, $I_{\lambda,\mu}$ satisfies the $(PS)_{\theta^-_{\lambda,\mu}}$-condition. Then, there exists $(u^2_{\lambda,\mu},v^2_{\lambda,\mu})\in\mathcal{H}$ such that up to subsequence $(u_n,v_n)\to(u^2_{\lambda,\mu},v^2_{\lambda,\mu})$ in $\mathcal{H}$. Moreover, $I_{\lambda,\mu}(u^2_{\lambda,\mu},v^2_{\lambda,\mu})=\theta^-_{\lambda,\mu}>0$ and $(u^2_{\lambda,\mu},v^2_{\lambda,\mu})\in\mathcal{N}^-_\lambda$, since $\mathcal{N}^-_\lambda$ is a closed set. Using the argument as applied in Theorem 5.4.1, one can easily obtain that $(u^2_{\lambda,\mu},v^2_{\lambda,\mu})$ is a nontrivial solution of system $(E_{\lambda,\mu})$ for $0<(\lambda\|f\|_\alpha)^{\frac{2}{2-r}}+(\mu\|g\|_\alpha)^{\frac{2}{2-r}}<\left(\frac{r}{2}\right)^{\frac{2}{2-r}}\Lambda_1$. □

Proof of Theorem 5.1.3: By Theorem 5.4.1 and Theorem 5.5.1, we obtain that for all λ, $\mu>0$ and $0<(\lambda\|f\|_\alpha)^{\frac{2}{2-r}}+(\mu\|g\|_\alpha)^{\frac{2}{2-r}}<\left(\frac{r}{2}\right)^{\frac{2}{2-r}}\Lambda_1$, system $(E_{\lambda,\mu})$ has two distinct solutions $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^+_\lambda$ and $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) \in \mathcal{N}^-_\lambda$, since $\mathcal{N}^+_\lambda \cap \mathcal{N}^-_\lambda = \phi$. Further, we show that the solutions $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ and $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ are not semi-trivial by using the same argument used in chapter 3. □

5.6 Conclusion

In this chapter, we establish the existence of at least two nontrivial solutions to a polyharmonic system involving critical nonlinearities with sign-changing weight functions with respect to the pair of parameters λ, μ belongs to a suitable subset of $\mathbb{R}^2$. Here, we extend the work of Shang and et. al [61] to the system of polyharmonic operators, in which they studied the multiplicity results for a polyharmonic equation. Moreover, One can also explore multiplicity results for Kirchhoff operator with critical nonlinearity.

6

Conclusion and future work

In this thesis, we explore the existence and multiplicity results for biharmonic/polyharmonic equations and biharmonic system with critical Choquard type nonlinearity with sign-changing weight functions. We observe that the Nehari manifold and fibering map analysis is one of the suitable techniques to deal with elliptic equations involving sign-changing weight functions. We notice that there was no work related to bihamonic operator with Choquard type nonlinearity. This is the starting point to work on this topic. We realise that one requires to study the minimzers for critical problems. So, the main contribution is to establish the minimizers for $S_{H,L}$ in case of biharmonic operator. With the help of these minimizers, we obtain the multiplicity results for biharmonic operator with Choquard type nonlinearity involving sign-changing weight functions. Later, we extend the multiplicity results for biharmonic systems.

In case of p-biharmonic operator, we are able to prove only the existence result in critical case. To obtain the multiplicity, one requires to study the minimizers for $S_{H,L}$, which is an open problem. In-fact, these minimizers are not known so far in case of p-Laplacian also.
In the future work, one can explore the multiplicity results for m-harmonic equations, Kirchhoff m-harmonic equation involving critical Choquard nonlinearity with sign-changing weight functions. To handle these problems, one can generalize the family of minimizers for $S_{H,L}$ in the case of m-harmonic operators in a similar manner as shown in chapter 2. The existence results for Kirchhoff biharmonic and polyharmonic operators with critical Choquard type nonlinearity are difficult to study. In fact, in the subcritical case, we need strong convergence to obtain a weak solution to the problem. Apart from that, one can examine the existence of infinite solutions for the critical Choquard equation involving polyharmonic operators.

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- **BSc,** 2011-2014, Sahu Jain Degree College, Mahatma Jyotiba Phule Rohilkhand University, Bareilly.

AWARDS/SCHOLARSHIP

- Secured 1024-rank in IIT-JAM in 2014.
- Secured 772-rank in IIT-JAM in 2015.
- Awarded fellowship in Ph.D. by Bennett University in 2017.

ARTICLES ACCEPTED/PUBLISHED

- Anu Rani, Sarika Goyal, Polyharmonic systems involving critical nonlinearities with sign-changing weight functions, Electron. J. Differential Equations, Vol. 2020 (2020), No. 118, pp. 1-25.
- Anu Rani, Sarika Goyal, Multiple solutions for biharmonic critical Choquard equation involving sign-changing weight functions, Topol. Methods Nonlinear Anal., DOI: 10.12775/TMNA.2021.025 .
- Anu Rani, Sarika Goyal, Biharmonic system with Hartree-type critical nonlinearity, (communicated).
- Anu Rani, Sarika Goyal, p-biharmonic equation involving Choquard nonlinearity, (communicated).

ACADEMIC VISITS

Attended:

- Workshop on Advance Training School On Functional Analysis For Teachers And Research Scholars conducted by IIT-Tirupati during December 11-23, 2017.
- IST on Partial Differential Equations: From Theory to Computation in IIT-Mandi during June 3-15, 2019.

- Presented a paper in International Conference on Mathematical Analysis and its Applications in South Asian University, Delhi, during December 14-16, 2019.
- Online Workshop on Essential Skills for Research Writing and Publishing during June 15-19, 2020 organized by Bennett University.
- Webinar Series on Advances in Mathematics-2020 organized by Tata institute of Social Sciences Tuljapur, during June 22-27, 2020.
- Virtual Mathematics conference (Virtual Math Fest 2020) during July 20-25, 2020.
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- Presented a contributed talk in the 4th BRICS Mathematics conference organised by IISER Thiruvananthapuram and BML Munjal University from December 7-10, 2021.
- Presented the poster in SIAM Conference on Analysis of Partial Differential Equations being held virtually in Berlin, Germany on March 14-18, 2022.