# Free Vibration Analysis of Functionally Graded Material Plates Resting on Elastic Foundation Using Dynamic Stiffness Method

Thesis submitted in fulfilment of the requirements for the Degree of

#### DOCTOR OF PHILOSOPHY

By

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I certify that the research work reported in the Ph.D. thesis entitled "Free Vibration Analysis of Functionally Graded Material Plates Resting on Elastic Foundation Using Dynamic Stiffness Method" submitted at Bennett University, Greater Noida, India is an authentic record of my work carried out under the supervision of Dr. Pawan Mishra and Prof. Vinayak Ranjan. I have not submitted this work elsewhere for any other degree or diploma. I am fully responsible for the contents of my Ph.D. Thesis.

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Material Plates Resting on Elastic Foundation Using Dynamic Stiffness Method" being

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I would like to dedicate this thesis to my loving parents

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## **Abstract**

In this present work, the dynamic stiffness method (DSM) is developed for free vibration analysis of functionally graded material (FGM) plates resting on Winkler and Pasternak elastic foundations. The material properties of the FGM plate vary continuously with according to different volume fraction laws such as power-law (P-FGM), sigmoid-law (S-FGM), and exponential-law (E-FGM) through the thickness direction of the FGM plate. A well-known, Classical plate theory (CPT) or Kirchhoff's plate theory is implemented along with the effect of physical neutral surface (PNS) to develop the kinematic variables or displacement components of the FGM plate. Hamilton's principle is applied to formulate the standard governing differential equation of motion for FGM plate resting on elastic foundation. A Levy-type solution technique is used to develop the dynamic stiffness matrix along with the displacement and force boundary conditions. The application of the Wittrick and Williams (W-W) algorithm is implemented to explain the complex behaviour of the obtained dynamic stiffness matrix and compute the accurate natural frequencies and mode shapes of the FGM plate resting on the elastic foundation. The obtained DSM natural frequencies results are compared with the available reported results and noticed that the DSM results are highly accurate with the existing results and can be further referred as a standard solution to compare the accuracy of various other analytical methods applied for analysing the free vibration response of FGM plate resting on elastic foundation. The effects of material and geometric property parameters (material gradient index, Young's modulus ratio, density ratio, aspect ratio, boundary conditions, and elastic modulus of foundations) on natural frequency are also examined. It is concluded that the present DSM method is efficient, reliable and accurate for examining the free vibration response of FGM plate resting on Winkler and Pasternak elastic foundations.

*Keywords:* Free vibration, Functionally graded material, Dynamic stiffness method, Physical neutral surface, Wittrick-Williams algorithm, Classical plate theory, Winkler and Pasternak elastic foundations.

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5.11 Fundamental frequency parameter  $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$  for SFSF rectangular 123 E-FGM plate with different combinations of  $(K_w, K_P)$  and h/a = 0.01.

# List of abbreviations and nomenclature

## **Abbreviations:**

DSM Dynamic stiffness method

FEM Finite element method

CPT Classical plate theory

FSDT First-order shear deformation theory

TSDT Third-order shear deformation theory

HSDT Higher-order shear deformation theory

DS Dynamic stiffness

MLPG Meshless Petrov-Galerkin

CUP Carrera's unified formulation

FGM Functionally graded material

PNS Physical neutral Surface

GDE Governing differential equation

DQM Differential quadrature method

W-W Wittrick-Williams

P-FGM Material property variation of FGM Plates according to power law

S-FGM Material property variation of FGM Plates according to sigmoid law

E-FGM Material property variation of FGM Plates according to exponential law

1D One dimensional

3D Three dimensional

SL Sinusoidal distributed load

UL Uniform load

## **Nomenclature:**

DSM Dynamic stiffness method

FEM Finite element method

a Length of the plate

b Width of the plate

h Thickness of the plate

p Material gradient index

*k* Volume fraction index

 $D_{FGM}$  Bending stiffness or Flexural rigidity of plate

*I*<sub>0</sub> Transverse inertia

J Number of frequencies lowers than the trial one

 $J_0$  Number of clamped natural frequencies

m Number of semi sin waves in the y directions

n Number of semi sin waves in the x direction

M Bending moment

Q Reduced stiffness

T Kinetic energy

U Potential Energy

 $U_{EM}$  Strain energy

V Shear force (transverse force)

K Dynamic stiffness matrix

 $K_W$  Winkler layer

 $K_P$  Shear layer

q Vertical displacement

u, v, w Displacements in the x, y, and z direction respectively

 $u_o, v_o$  Middle-surface in-plane displacement components in x, y direction

respectively

x, y, z Coordinate system

 $z_0$  Location of middle surface

 $z_{ns}$  Location of Physical neutral surface

 $F_x$  Total axial force in a particular x, y direction respectively

 $D_{FGM}$  Flexural rigidity of the FGM plate

 $\varepsilon$ ,  $\gamma$  Normal and shear Strain

ν Poisson ratio

 $\rho$  Density

Ø Rotation

E Young's modulus

 $\alpha$  Half sin wave

## **Super and Subscripts:**

c Ceramic constituents of FGM plate

m Metallic constituents of FGM plate

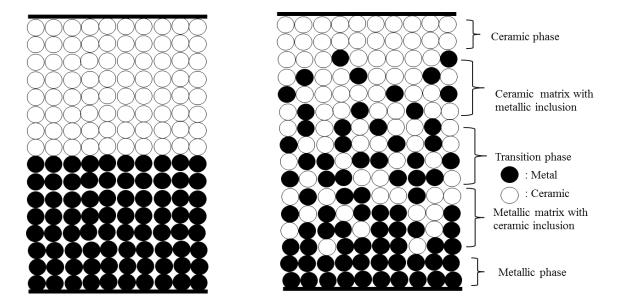
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## **CHAPTER 1**

## Introduction

This thesis deals with analyzing the free vibration behaviour of functionally graded material (FGM) plates with elastic foundations using the dynamic stiffness method (DSM). Functionally graded materials (FGM) are inhomogeneous composite materials whose material properties continuously vary from one layer to another in the required directions [1, 2]. In the last decade, various authors have shown interest in the expeditious development of FGM and found impending applications in the different engineering fields. The composite materials are generally made from two or more material constituents, generating an improved response, which is generally impossible to attain with single material constituents. In the design and manufacturing of the structural components of functionally graded material plates, the preferred property variation is through the thickness or transverse direction of the plate. Generally, functionally graded material plates are formed from ceramic and metal. In contrast, ceramic constituents resist high temperature, and metal constituents resist high fracture and enhance the strength of the FGM plates [3, 4]. Due to high stiffness and strengthto-weight ratios, the FGM plates are favorably applied as lightweight components bearing heavy loads. Taking these advantages, structural components of FGM plates are extensively applied in various engineering fields [5, 6].

The comparison of traditional composite material and functionally graded material (FGM) at the microscopically level can be schematically represented in Fig.1.1. It is noticed that the material gradation detail description in the FGM is generally very difficult to describe except for the distribution of volume fraction [7]. In each constituents of the FGM, the distribution of volume fraction is generally varied in the gradation direction of the material. Note that the constituents variation within the FGM results in the mechanical properties variation within the given material. Researchers and scientists have implemented several mathematical models to show this material property variation, and these mathematical models are explained in Section 1.2.



- (i) Traditional Composite Material
- (ii) Functionally graded Material

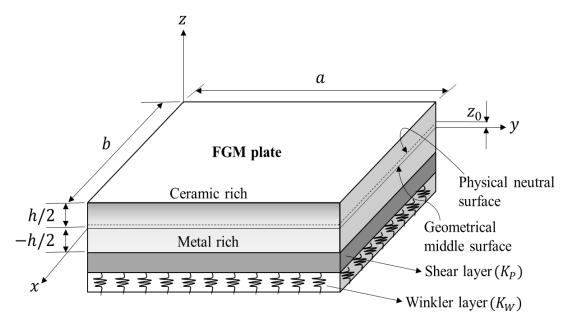
**Fig. 1.1:** Schematic representations of microstructures in (i) traditional composites and (ii) functionally graded material

The main focus of this present thesis is to develop the dynamic stiffness method to analyse the free vibration characterization for rectangular FGM plates with elastic foundations. The proposed dynamic stiffness method is considered a more elegant and powerful alternative than the conventional finite element method [9] and generates more accurate results independent of the number of elements used in the investigation. For the implementation of DSM, the frequency-dependent dynamic stiffness (DS) matrix is considered the primary building block where the DS matrix is a function of the mass and stiffness property of the given element. It is well-known that the formulation of dynamic stiffness effectively generates the exact solution because the developments of elemental dynamic stiffness are obtained from the exact solution of a generalized governing differential equation (GDE) [10]. Thus, the DSM element is based on the exact shape function and has better modeling capability than the finite element method (FEM). Therefore, it is very accurate and computationally efficient. Unlike the FEM, there is a rare chance of discretization error in the DSM approach.

Moreover, this method can also predict an infinite number of free vibration modes by means of a finite number of coordinates. Thus, the dynamic stiffness method is believed to be superior to other approximate methods such as finite element and finite difference methods. As DSM is based on a frequency-dependent shape function derived from the exact solutions

of the (GDE), it can be treated as an exact method, especially in the computation of higher vibration modes of structures. Therefore, properly implementing this DSM approach results in a very high degree of accuracy in the frequency computations.

The literature survey found that very limited work adopted the DSM approach to examine the free vibration behaviour of functionally graded plates embedded on elastic foundations [11]. Also, the application of the Wittrick-William algorithm in the DSM formulations for free vibration of FGM plates is scantly reported in the literature. Hence, this present thesis develops the DSM with Wittrick and Williams algorithm to analyse the free vibration behaviour of functionally graded plates resting on elastic foundations. The illustration of the FGM plate resting on elastic foundation is represented in Fig. 1.2. The material property variation for free vibration analysis of the FGM plate is derived by implementing the three different property variations laws such as power-law, sigmoid-law, and exponential-law.



**Fig. 1.2:** Schematic of the FGM plate supported by elastic foundation with the representation of the geometrical middle surface and physical neutral surface.

In the present DSM formulation of the FGM plate, the (CPT) along with the concept of physical neutral surface (PNS) [12, 13] of the plate, is used to derive the dynamic stiffness (DS) matrix. Here, the physical neutral surface of FGM plates can be considered a conceptual plane similar to the definition of the neutral plane in Euler-Bernoulli beam theory. For the cases of bi-metallic or composite beams, it can be noted that the neutral plane does not pass through the centroid of the beam cross-section. Similarly, the neutral plane in a functionally

graded material plate does not coincide with the geometrical middle surface of the FGM plate, and its exact location depends on the volume fractions and gradation of the constituent materials within the FGM. Once the DS matrices of the individual plate elements are obtained, they are suitably assembled to form the global DS matrix of the entire plate.

Furthermore, the global dynamic stiffness matrix is solved in the final step using the well-known Wittrick-Williams algorithm [14] to compute the natural frequencies. The frequency results obtained from the DSM methodology are validated by those reported in the literature. In this study, it has been shown that these results are most accurate, and they can be further implemented as a benchmark solution for validation purposes. Also, some inaccurate reported results have been pointed out, and the possible reasons for these inaccuracies are discussed. Additionally, different tables and graphs are presented to highlight the effect of different plate parameters on the natural frequency results of these FGM plates.

The remaining part of this chapter can be described as follows. Section 1.1 described the background and application of the present study. Section 1.2 explained the mathematical model for material property variation of the FGM plate. Section 1.3 described the elastic foundation phenomena with applied Winkler and Pasternak elastic medium. Section 1.4 defined the motivation of the present work. Sections 1.5 and 1.6 well-define the objectives and contributions of the present study, respectively. Continuously, the limitations of the scope of the present thesis are highlighted in Section 1.7. In the end, Section 1.8 reported the outline of the remaining thesis.

## 1.1 Importance and application of functionally graded materials

The conceptual idea of the functionally graded material was developed in 1984 by the Japanese scientist, whereas the first implementation theory of proposed FGM was given to develop the thermal barrier material under working on a space project [15]. However, similar materials have already existed due to specific features of living tissues (functional gradation). Some examples of natural FGM include bamboo, human skin, bones, etc. [16]. Excitingly, based on our skin depth (property) and body position, it offers certain tangible, elastic, and rigid qualities. On the other side, the artificial or human-made FGM has two isotropic material phase constituents where the material volume fraction index continuously varies through the material body. While these material constituents generally formed a phase of a

metallic constituent, such as engineering aluminum alloys of copper, tungsten, steel, titanium, magnesium, etc., and a phase of ceramic constituent.

In recent years, the applications of functionally graded materials have been highly preferred over conventional composite materials because of their excellent features in severe (required) working/operating conditions. As explained earlier, the upper layer of the FGM comprises ceramic rich for proving excellent performance in high thermal working conditions, whereas the lower layer of the FGM comprises metal-rich, which decreases the unexpected transition of the thermal expansion coefficients, increases load carry capacity, and resist thermal corrosion. Primarily, the functionally graded materials were manufactured for the aerospace structural components to resist high-temperature fluctuation. However, in recent years, the various potential applications of FGM are developed, and as represented in Fig. 1.3 along with the continuous growth is produced because the FGM is generally personalized as per the requirement and practical design purpose.

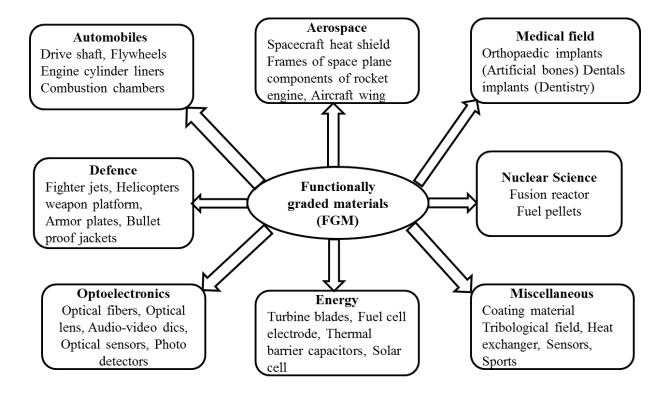


Fig. 1.3: Application of functionally graded material (FGM) in different potential fields.

As explained earlier, the functionally graded material was used first time for the space field, and by taking its advantages, its use continuously expanded in heat exchanger tubes, fusion reactors plasma-facing, flywheel, spacecraft heat shields, solar energy generators, biomedical

implants, chemical plants combustion chamber, etc. [17]. Functionally graded material can be considered as a thermal barrier (resist high temperature) hence it can be implemented for insulation of rocket engine parts, shielding spacecraft heat, and resisting heat in the combustion chamber. FGM, together with fibers of TiAl/SiC, are applied in the missile leading side, rocket nozzle, space shuttle, nose caps, spacecraft truss structure, and heat exchanger panel. FGM surrounded with carbon nanotube presents excellent mechanical features, high abrasion resistance, flexural strength, toughness, and hardness under both high and low-temperature fields (regions), so it can be stable under thermal conditions [17, 18]. The lightweight FGM is used in defense and military areas such as fighter jet and lightweight helicopter components, barrier materials, bulletproof jackets, defense tanks, weapon platforms, and armor suits [6, 17]. Certain of these functionally graded materials have chemical inertness with good damping properties. Hence, it can be applied to stabilizer components, aircraft wings, gas turbine engines, cryogenic propellant tank rotor blade, fuselage tank, and nozzle and compressor parts of fighter planes [17, 19]. Graphite/epoxy FGM, carbon/glass FGM, Al/SiC FGM, and Glass/epoxy FGM are used to make composite sonar domes piping systems, diving cylinders and cylindrical pressure hulls, and propulsion shafts, there are some important components of military fields (submarine) [6, 17, 19].

In the last decade, functionally graded material has been gaining wide attention in the application of biomedical fields such as implantation of teeth and replacement of the knee, hip, and joint. FGM applied for dental implantation can show optimized mechanical responses, which can help to reach the required physical characteristics of Osseointegration and biocompatibility [19-21]. Generally, in dental implants, the titanium-based alloy with hydroxyapatite/collagen bioactive (HAP/Col) is correctly mixed to get the required FGM response [22-24]. In several orthopedic implantations such as artificial hip replacement, knee joints, and shoulder replacement, the biopolymer of functionally graded material with a high-density polyethylene coating is implemented [22, 24, 25]. Under the replacement of hip and knee joints, Tawakol [26, 27], Tawakol and Bondok [28], and Jassir et al. [29] advised that by applying the FGM in the place of CoCrMo material and titanium alloy the considerable amount of reduction in the stress distribution of the tibia tray, stem and bone cement is noticed.

Theoretical and mathematical observation determined that optoelectronic devices with functionally graded material have much better response/characteristics than traditional optoelectronic devices. For illustration, the diffusion length, energetic band gap, and the

refractive index modulation can be adjusted and absorption capabilities with generated efficiency can be improved. Hence, FGM are widely implemented in anti-reflective layers, optical sensors, semiconductors devices, cellular phones, solar cells, computer circuit boards, optical lenses and fibers, photo-detector, etc. FGM can be approaching materials for recent devices of optoelectronic such as threshold current edge lasers (GRINSCH) and tunable photodetectors [30-32].

Under the consideration of automobile industries and their applications, FGM, along with Al/SiC, is generally applied in different automotive structures like a driving shaft, racing car brakes, engine cylinder liner, and flywheel, etc. FGM with SiCw/aluminium-alloy and Al/C are usually applied to form diesel engine pistons and leaf springs, respectively. FGM with coating material decrease the heat losses from the engine exhaust panel or system such as exhaust headers, turbocharger, down pipes and tailpipes, and exhaust manifolds, thus decreasing/minimizing the coolant utilization [33]. FGM coated with TiAl/SiC resists the high temperature in turbine gas engine assembly under working 40,000 rpm turbine wheel blade. FGM with the coating is also implemented to produce most cutting, forming, forging, and machine tools. Some basic examples of FGM are associated with anti-abrasion sports equipment based on stress relaxation, such as baseball cleats, sports shoes, tennis rackets, racing bicycle frames, etc.

**Table 1.1:** Potential application of various functionally graded materials with their significant properties [6].

FGM	Significant property	Application
Al2O3/Al-alloy	Thermal barrier and corrosive resistance	Rotary launchers, wings, Rocket nozzle, engine casting
E-glass/Epoxy	Hardness and damping resistance	Brake rotors, solar domes, Composite piping systems
Al/SiC	Hardness and toughness	Combustion chambers, racing car breaks, Engine cylinder liners, Flywheels
TiAl/SiC, SiC/C	Temperature and shock-resistant coatings	Wind turbine blades, rocket nozzle, heat exchange panels, turbine wheel blades, solar panels, reflectors, spacecraft truss structure,
Carbon/Epoxy	Lightweight and good damping properties	Components of helicopter such as engine parts, doors, heat exchanger panels, landing gear
SiCw/Al-alloy	Hardness and toughness, thermal resistance, chemical inertness,	A racing bicycle, vehicle frames, diesel engine pistons, storage cylinders
Graphite/Epoxy	Reduces thermal distortions, high strength to stiffness ratio and good 7	Satellite antennas, Sonar domes, cylindrical pressure hulls, space telescopes, cryogenic

	radiation resistant	tanks
Al/Al2O3,	Thermal, wear and tear, corrosive	Artificial bones, cutting tools, forming and
WC/Co,	resistance	machine tools,
BaTiO3/Si	Control of signal loss at high frequency	Dielectric motors
Al alloy/Carbon nanotubes (CNT)	Lightweight and high stiffness	MRI scanner spares, artificial ligaments, eyeglass frames, musical instruments, dentistry parts

There are some other uses of FGM such as cryogenic tubes, fire-fighting air bottles, fire retardant doors, sensors, solar cells, high efficient photodetector, window glasses, camera tripods, MRI scanner parts, X-ray tables, laptop cases, wind turbine blades, titanium watches, vessels, and fuel tanks, pressure vessels of the automobile, helmets, razor blades, eyeglass frames, etc. [34, 35, 36]. The typical applications of functionally graded materials with their functional properties in tabular form in Table 1.1.

#### 1.2 Mathematical models for FGM property variations

As mentioned earlier, the FGM plate is formed by continuously varying the volume fraction of the given constituent phases in the transverse direction. The constituents variation through the transverse direction also affects the mechanical property variation of the FGM plate. Usually, three standardized mathematical models are used to explain the variation of the material property of the FGM plate through the transverse direction [37-39].

## 1.2.1 Power-law (P-FGM)

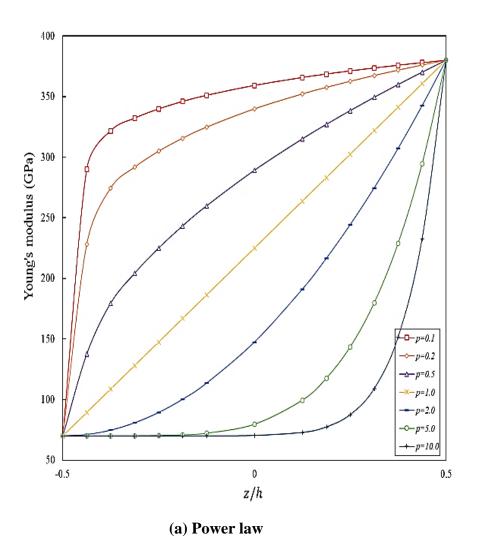
A well-known mathematical model is used to describe the material property variation based on the mixture rule [40] and is known as power-law (P-FGM). It is widely implemented in the published literature to investigate the different characterizations of the FGM structures. The standard expression of power-law can be given as

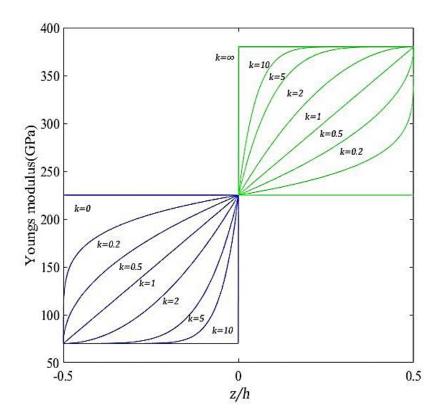
$$E(z) = V_c(z)E_c + V_m(z)E_m$$

$$= (E_c - E_m) \frac{1}{2} \left(1 + \frac{2z}{h}\right)^p + E_m,$$
(1.1)

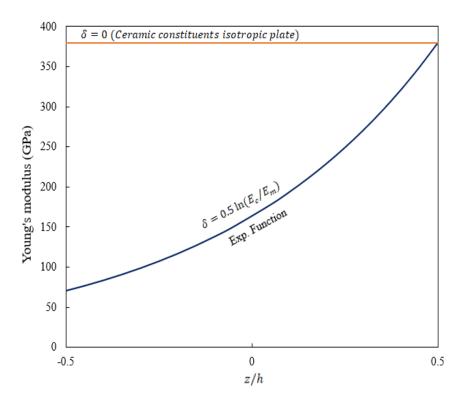
where (p) represents the non-negative material parameter and is called the material gradient index, which describes the volume fraction of given material. The mixture rule is applied to determining the material property given by Eq. (1.1), and the material property of metal and ceramic are represented by  $E_m$  and  $E_c$ , respectively.

The material property variation of the power-law FGM plate is indicated by density  $\rho(z)$  and Young's modulus E(z), respectively, and represented by Eq. (1.1).





## (b) Sigmoid law



(c) Exponential law 10

**Fig. 1.4:** The variation of Young's elasticity of the FGM plate along the thickness direction (*z* direction; see Fig. 3.1 in Chapter 3) as per (a) power law, and (b) sigmoid law and (c) exponential law. The material properties for the constituent materials of the FGM plate considered here are shown in Table 3.1 of Chapter 3.

Fig. 1.4 (a) shows the variation of material property (Young's modulus E(z) and density  $\rho(z)$ ) of the P-FGM plate. Here at the material gradient index (p=0), the effect of the given material property does not change in the thickness direction of the P-FGM plate, and it is considered a homogenous isotropic plate in that direction. By applying the significant value of material gradient index (p), the P-FGM plate acts as a bi-material in nature where the plate top side is ceramic-rich and the bottom part is metal-rich, respectively, as determined in Fig.1.4 (a).

#### 1.2.2 Sigmoid law (S-FGM):

According to the sigmoidal law function [37-39], the expression of material property variation in the transverse direction of the plate can be given by

$$P_{1}(z) = \frac{(P_{m} - P_{c})}{2} \left(\frac{1}{2} - \frac{z}{h}\right)^{k} + P_{c} \quad \text{for } 0 \le z \le h/2$$

$$P_{2}(z) = \frac{(P_{c} - P_{m})}{2} \left(\frac{1}{2} + \frac{z}{h}\right)^{k} + P_{m} \quad \text{for } -h/2 \le z \le 0$$
(1.2)

The non-negative material parameter is represented by (k) and is known as the sigmoid law index (or volume fraction index), which defines the material volume fraction. The two halves' variation of material properties of S-FGM are represented by Young's modulus  $E_1(z)$ ,  $E_2(z)$  and density,  $\rho_1(z)$ ,  $\rho_2(z)$ , respectively, and can be described by Eq. (1.2).

The variation of material properties of the S-FGM plate is shown in Fig 1.4 (b). For the sigmoid law index, k = 0, the distribution of the material property of the S-FGM plate does not change in the transverse direction and is called a transversely isotropic homogeneous plate. When the value of the sigmoid law index (k) is very high, the FGM plate behaves as a bi-material in nature where the upper layer of the plate is highly rich in ceramic, and the bottom layer is highly rich in metallic, as observed in Fig.1.4 (b).

## 1.2.2 Exponential law (E-FGM):

As per the exponential-law function [41,42], the variation of material property along the transverse direction is given by

$$P(z) = P_c e^{-\delta \left(1 - \frac{2z}{h}\right)} \quad \text{with } \delta = \frac{1}{2} \ln \left(\frac{P_c}{P_m}\right)$$
 (1.3)

where P(z) represents the material property through the transverse direction of the FGM plate and  $P_c$  and  $P_m$  indicate the value of material properties of ceramic and metal constituents, respectively. It is noticed from Eq. (1.3) that at  $\delta = 0$ , the value of  $P_c = P_m$ , which is considered an E-FGM plate, behaves like a homogenous isotropic plate with a ceramic constituent. The variation of Young's modulus through the transverse direction of the plate as described in Eq. (1.3) is shown in Fig.1.4 (c).

In this study, the Poisson's ratios ( $\nu$ ) property variation along the transverse direction of the plate is not considered. Because the FGM plate is considered thin, the variation of the Poisson's ratios ( $\nu$ ) in the transverse direction may become negligible [37-42].

#### 1.3 Elastic foundation model

A Winkler model [43] is defined as an approximated series of closely spaced mutually independent vertical linear springs, where the stiffness of the spring is assumed to be equivalent to the elastic modulus of the foundation. Pasternak has improved the Winkler model [44] where the shear interaction is applied between the spring elements. This is achieved by connecting the ends of the springs to the plate that only undergoes transverse shear deformation, as shown in Fig. 1. The advantage of this model is that it considers both transverse shear deformation and normal pressure of the surrounding elastic medium. Considering this advantage, the free vibration behavior of the foundation structure interaction is widely described by the Pasternak model [45-47].

Reaction force or normal force [48] of the foundation can be expressed as

$$q_{\text{Winkler}} = k_W w$$

$$q_{\text{Pasternak}} = k_W w - k_P \nabla^2 w \tag{1.4}$$

where q represents the reaction force, or normal force of the foundation and w represent the vertical displacement,  $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ ,  $k_W$  and  $k_P$  represent the Winkler and Pasternak elastic modulus, respectively.

## 1.4 Motivation of the present study

The present research work is highly motivated by structural design problems in the engineering field's conjunction with the natural vibration behaviour analysis of the FGM plates resting on elastic foundations. As mentioned earlier, due to some excellent features or advantages of FGM over conventional composites materials, it has created great interest among many researchers and scientists working in the design of structural engineering fields. The engineering structures based on the FGM plates resting on elastic foundations are extensively used in aerospace, automotive, civil, and marine engineering design, amongst many other industrial fields such as foundations for tanks, railroad tracks, nuclear reactors, impact-type machines and turbo generator [49-50]

## 1.5 Objectives of the present work

In the presented literature survey, it is observed that several works (e.g., works in [51-267] have explained the free vibration response of FGM plates. On the other hand, these works implemented various mathematical methodologies to extract the free vibration frequency results of the FGM plates. To the author's best knowledge, significantly less work has been reported where applied DSM methodology along with the Wittrick-Williams algorithm, to investigate the free vibration response of FGM plates resting on elastic foundations. It is already mentioned that the DSM is a highly efficient, robust, and reliable method and provides a very degree of accurate computation frequency results.

In this reference, the main objectives of the present study are explained below.

- To develop a dynamic stiffness method and the application of The Wittrick-Williams algorithm to investigate the natural frequencies of the FGM plates resting on Winkler-Pasternak elastic foundations.
  - (a) For Levy-type boundary conditions (two opposite edges are simply supported)
  - (b) For different volume fraction laws (power-law, sigmoid-law, exponential-law)
  - (c) For different elastic foundation parameters (Winkler-Pasternak)
  - (d) For different material and geometric properties such as Young's modulus ratio, density ratio, aspect ratio, and volume fraction index

- To compare and validate the DSM results with the published literature results.
- To generate a new set of free vibration frequency results for FGM plate resting on Winkler and Pasternak elastic foundations, which can be used as a benchmark solution.

### 1.6 Limitations and scope of the thesis

The present thesis analyzes the free vibration characterization in the thickness direction of the thin FGM plate resting on elastic foundations. The variations of material property through the thickness direction continuously vary using three different mathematical models such as power law, sigmoid law, and exponential law in terms of volume fraction of the given constituents. The displacement components of the FGM plate are described according to CPT. Therefore, the shear deformation effect of the plate is neglected, and this research is valid only for thin FGM plates. The Levy-type (displacement and force) boundary condition is implemented where two contrary edges of the plate are simply supported, and the other two are arbitrary conditions (free, clamped, and simply supported). Note that, in this study, the variation of the Poisson's ratios (v) property along the transverse direction of the plate is not considered. As the FGM plate is deliberated to be thin (as per classical plate theory), the transverse variation of the Poisson's ratios (v) may become negligible. Due to thin FGM plate geometry, the variation of material properties (Young's modulus ratio and density ratio) continuously vary in the transverse direction, and the in-plane property variations become very small and can be ignored. Under the present study's limitations and relevant scopes, the thesis main contributions can be described as follows.

### 1.7 Contributions of the thesis

This present study has formulated a dynamic stiffness method for analyzing the natural vibration response of a thin FGM rectangular plate resting on the Winkler and Pasternak elastic foundation. Classical plate theory (CPT) is implemented along with the effect of physical neutral surface (PNS) to develop the kinematic variable or displacement component of the FGM plate. Hamilton's principle is applied to formulate the standard governing differential equation of motion for an FGM plate resting on elastic foundation. The Wittrick and Williams (W-W) algorithm is implemented to solve the complex behaviour of the obtained dynamic stiffness (DS) matrix and compute the accurate natural frequencies and

mode shapes of the FGM plate resting on the elastic foundation. The main contribution of the present study can be concise as follows:

- The dynamic stiffness method is formulated to evaluate the response of free vibration behaviour of P-FGM, S-FGM, and E-FGM plates supported on Winkler- Pasternak elastic foundation.
- The Wittrick-Williams algorithm is implemented to extract the accurate natural frequencies of the FGM plate.
- Instead of a geometrical mid surface, a physical neutral surface is applied to model the FGM plate.
- The natural frequencies of the FGM plates are validated and compared with the
  published results to prove that the present obtained results are highly accurate. It can
  be used further to examine the accuracy of different methods.
- A new set of natural frequencies of FGM plates resting on Winkler-Pasternak elastic foundation are obtained and reported for different volume fraction index, Winkler and Pasternak elastic modulus, modulus ratio, density ratio, and aspect ratio.

The following thesis chapters have reported all the above-explained works in detail.

It is motivating to observe that the present work extracted frequency results using DSM for thin rectangular FGM plates resting on the elastic foundation are highly accurate compared with existing reported results in the literature, and therefore it can be applied as a benchmark standard solution for the future validation purposes.

#### 1.8 Outline of the thesis

The structure of the remaining thesis is explained as follows:

### i. Chapter 2

Chapter 2 explained the literature review of the present work, where the first described the development and qualitative contribution of the dynamic stiffness method. The second part explained existing literature based on free vibration analysis of rectangular FGM plates without elastic foundations and finally highlighted important literature related to free vibration analysis of rectangular FGM plates with elastic foundations.

### ii. Chapter 3

Chapter 3 developed DSM and Wittrick-Williams algorithm to compute the natural vibration response of FGM plate resting on elastic foundations where material property variation is explained by using power law (P-FGM) function.

### iii. Chapter 4

Chapter explained the formulation of dynamic stiffness method for analysing the free vibration response of sigmoid functionally graded (S-FGM) rectangular plate resting on elastic foundations.

### iv. Chapter 5

Chapter 5 described the formulation of DSM methodology to study the free vibration characterization of exponential functionally graded (E-FGM) plates resting on elastic foundations.

### v. Chapter 6

In Chapter 6, the conclusions based on the investigation of overall thesis have been highlighted.

### vi. Chapter 7

In Chapter 7, the scope of the future work of the present thesis has been highlighted

### **CHAPTER-2**

### **Literature Review**

This section of the thesis first explained the qualitative contribution of the dynamic stiffness method (DSM), starting from the earliest useful work to the latest pioneering developments in various structural components in different engineering fields. Starting with describing the significant contribution in analyzing the free vibration behaviour response of plate structure based on different plate theories using the dynamic stiffness method. In the next part, literature surveys based on free vibration analysis of FGM plate structures, different analytical approaches, and various plate theories are reported. In the end, some significant literature and characterization of free vibration response of FGM plate resting on Winkler and Pasternak elastic foundations, with different numerical method and different plate theories, are also discussed.

### 2.1 Literature review based on the dynamic stiffness method

In the field of plate structures analysis, first time the DSM was applied to investigate the free vibration behavior of a continuous plate by the authors Veletsos and Newmark [51], where authors extracted the natural frequency of the hinged supported (two opposite sides) plate. Later, in the early seventies, extended the formulation concept of DSM by Wittrick-Williams [14] to analyse the buckling and vibration response of isotropic and anisotropic plate structures and applied the classical plate theory. Later, Wittrick and Williams [52] developed dynamic stiffness method based on lumped mass system to analyse the undamped natural frequency of the continuous and uniform linearly elastic plane frame structures subjected to static loading. They developed general-purpose computer programs that can extract the free vibration frequencies of an infinite variety of possible structures. Further, Wittrick and Williams established a software program called VISPASA [53-55] to employ DSM for the plate structure problem. After that, that code, in the developing stage, took different coding names like VICON [56], PASCO [57, 58], and VICONOPT [59, 60], etc. During these research works, an efficient, reliable, and robust algorithm well-known as Wittrick and Williams algorithm was implemented to carry out the natural vibration frequency of plate

elements or structures. It ensures from the applied algorithm that no natural frequencies are missed in the given complex structures (for brief explanations of W-W algorithm, refer to Section 3.2.8 of Chapter 3). Kirchhoff's plate theory was used in the earlier mentioned VICON software program for composite plate structures. Later, the advanced version of VICONOPT software programs was applied by Anderson and Kennedy [60], where the shear deformation effect was considered in their numerical modeling. After that, Langley et al. [61] applied a 1D modified Fourier series and dynamic stiffness method for analyzing the free vibration problems for plates under arbitrary boundary conditions. Further, continuing his work, Langley [62] developed the dynamic stiffness method to analyse rectangular aircraft panels' free and forced vibration problems under simply supported boundary conditions. The proposed method was implemented for a single, six, and infinite panel row. Later, Leung [63] dynamic stiffness method, along with the Kantorovich method, reduced spatial discretization errors based on the amplitude of applied force and exponentially harmonic excitations to study the natural vibration response of thin-walled structures.

In the early nineties, a comprehensive historical literature survey based on the dynamic stiffness method to analyze the vibration behavior of plate structure [64]. Later, Banerjee and co-authors developed DSM for its wider range of applications in various simple and complex structural components. As explained earlier, the conceptual idea of DSM was developed a long year back, and most of the authors applied the same for different structural problems. On the other hand, a general mathematical approach for formulating dynamic stiffness (DS) matrix based on applying different structural problems was not done. After that, in the same direction, Banerjee [11] developed a general mathematical process to formulate the dynamic stiffness matrix for a structural element. In that work, an accurate mathematical procedure to formulate the efficient dynamic stiffness matrix and this approach subsequently save the computational time under applied explicit mathematical expression instead of a numerical approach for the DSM. Further, continuing his work, Banerjee and co-authors [65-70] extensively used DSM for analyzing the beam structures vibration response using Euler-Bernoulli, Timoshenko theory, and considered coupled beam theory. Later, Banerjee and coauthors extended the concept of DSM for simple beam structural elements to plate structural elements and other complex structural components. They reported various research articles [11, 71-81] based on dynamic stiffness development and analyzed the vibration characterization for isotropic and composite plate structures with various boundary (edge) conditions. Afterward, Ghorbel et al. [82, 83] also implemented DSM for investing the inplane and out-of-plane vibration behavior of orthotropic plate structures. Leung [84] applied the dynamic stiffness method based on harmonic oscillation to study the free vibration and dynamic buckling effect to reduce the spatial discretisation error of the thin-walled structures. Yu and Roesset [85] analytically formulated the dynamic stiffness method based on the frequency domain to analyze the dynamic behavior of the free vibration response of linear structure members and distributed mass. This formulation considers not only the distributed mass but also the wave propagation impact on each element. Lillico et al. [86] formulated the dynamic stiffness method and quadratic functions to study the free vibration modal analysis of aircraft composite wings. Leung and Zeng [87] applied the dynamic stiffness method along with the Kantorovich method to analyze the uniform, non-uniform, straight, curved, damped, and undamped free vibration problems of skeletal structures. Moon and Choi [88] used the dynamic stiffness method to study space frame structure's free and forced vibration problem. The obtained numerical results were validated with reported by experimental results and the finite element method Zhaung et al. [89] implemented the dynamic stiffness method along with the superposition and projection method to examine the free and forced harmonic vibration response of 3D coupled thin plate structures under-considered flexural and in-plane both vibrations. Popkov et al. [90] used the Fourier series in conjunction with the dynamic stiffness method with point nodes to analyze the free vibration response of rectangular plates under different boundary conditions. Danilovic et al. [91] applied Gorman's superposition method along with the Projection method and formulated DSM for analyzing the free vibration behavior of rectangular plates. Fazzolari et al. [92] formulated an exact dynamic stiffness method based on higher-order deformation theory (HSDT) to study the natural vibration response of plate assemblies. Hamilton's principle was applied to obtain the GDE of motion. Banerjee et al. [93] developed the dynamic stiffness method for the most general case based on the bi harmonic motion equations to investigate the free vibration response of rectangular plates. For deriving the dynamic stiffness matrix of plate where consider all the sides free boundary conditions and possible to generate exact free vibration solution of plate and plate structures under any boundary conditions. Fazzolari [94] employed the dynamic stiffness method based on HSDT to compute the natural frequency of the cross-ply laminated cylindrical and spherical shells. Thinh et al. [95] developed dynamic stiffness (DS) matrix for a continuous element of laminated cylindrical shell using the dynamic stiffness method based on an analytical solution for analyzing the free vibration response of the shell. the obtained frequency results were compared with the reported finite element method (FEM) frequency

results. Pagani et al. [96] applied Carrera Unified Formulation (CUF) and dynamic stiffness method based on refined beam theory to investigate the free vibration characterization of the composite plate. The accuracy of DSM methodology was verified by the reported FEM solution from the experimental test and commercial code of MSC/Nastran. Further, Pagani et al. [97] applied the same methodology to analyze thin-walled structures' free vibration behavior. Liu et al. [98] applied 1D modified Fourier series and DSM for free vibration modal analysis of plane elastodynamic behavior under-considered plane stress and strain assumption. Popkov et al. [99] formulated a dynamic stiffness method based on Mindlin plate theory to study the effect of rotatory inertia and shear deformation to study the free vibration response of FGM rectangular plates. Ghorbel et al. [100] implemented Kirchhoff's plate theory and dynamic stiffness method to compute the free vibration behavior of the rectangular plate under free edge boundary conditions. Kumar et al. [101,102] used the dynamic stiffness method based on Classical plate theory with Levy type solution to study the natural vibration behavior of rectangular and stepped FGM plates. Ali et al. [103] applied the same methodology with Kirchhoff plate theory for analyzing the natural vibration problems of FGM rectangular plates. Kumar et al. [104] used the dynamic stiffness method based on FSDT to investigate the natural vibration behavior of rectangular stepped FGM plates with applied Levy-type boundary conditions. Liu et al. [105] implemented a modified Fourier series with the dynamic stiffness method to study the modal analysis and damped dynamic membrane assemblies' analyses under different boundary conditions. Nguyen et al. [106] employed a dynamic stiffness method based on FSDT shell theory and applied Coriolis forces, centripetal forces, and rotary inertia forces to extract the free vibration response of laminated composite conical shells. Tian et al. [107] used the dynamic stiffness method based on Flugge shell theory to study the free and forced vibration characterization of conical, cylindrical shells under different boundary conditions. Mochinda et al. [108] applied the dynamic stiffness method and Gorman's superposition method along with the Rayleigh-Ritz method for analyzing the natural frequency and mode shapes of the structural elements of the plates. Khlifi et al. [109] formulated the dynamic stiffness method based on Kirchhoff's-Love thin plate theory and a novel Levy series solution to compute the natural frequency of the rectangular plates. Later, various researchers [110-115] formulated the dynamic stiffness method for analyzing the vibration response of homogeneous isotropic and composite plates under a different set of arbitrary conditions.

### 2.2 literature review on free vibration analysis of FGM plate structures

In this subsection of the thesis, some relevant and important literature survey is reviewed to analyze the free vibration behavior of FGM plate structures with or without Winkler and Pasternak elastic foundations based on different plate theories and analytical methods. First explained the literature review based on free vibration analysis of FGM plate structure without elastic foundation. Later, reported the literature review based on the free vibration behavior of FGM plate structure with elastic foundation.

## 2.2.1 Literature review based on free vibration analysis of FGM plate structures without elastic foundations

The literature on free vibration characterization behavior of functionally graded materials (FGM) plate structures applying different plate theories without elastic foundations is rich. Here, some of the important, relevant published works are highlighted. For example, Woo et al. [116] solved governing differential equations using a series method and applied CPT to investigate the high-frequency amplitude of the FGM plate with applied different boundary conditions. Ng et al. [117] used the assumed mode technique and Hamilton's principle to solve the free vibration and parametric behavior of rectangular FGM plates under applied CPT with in-plane loading conditions. Praveen et al. [118] used the finite element method with von Karman plate theory for analyzing the free vibration behavior and parametric study of the FGM plate. Huang and Shen [119] used TSDT with the perturbation method to analyze FGM plates' free nonlinear vibration and dynamic behavior under thermal environment conditions. It is obtained from the reported results that the volume fraction distribution and temperature distribution have a significant effect on free nonlinear vibration and the dynamic behavior of FGM plates. Allahverdizadeh et al. [120] used von-Karman equations with a developed semi-analytical technique based on CPL to analyze the circular FGM plate's free and forced vibration. Kim [121] applied the Rayleigh-Ritz solution method with TSDT to study the natural vibration response of initially stressed rectangular FGM plates under thermal conditions. Thermal loads were applied in two conditions: in the first condition temperature value was implemented on the top surface and the other value of temperatures on the bottom surface. In the second condition, heat flow was applied on top to the bottom surface with the help of a single prescribed temperature. Baferani et al. [122] used CPT with Navier solution to characterize the free vibration behavior for different boundary conditions of the FGM plate. Yuda and Xiaoguang [123] used the Galerkin method based on CPT with applied in-plane excitation to analyze the stability and parametric vibration of the FGM plates. Reddy et al. [35] used the finite element method and FSDT to study the free vibration response with the bending and stretching effect of FGM circular and annular plates. Later, Reddy [36] presented the finite element model and theoretical formulation based on TSDT. Yin et al. [124] used B-spline (NURBS) function based on CPT to analyze the isogeometric characterization of the FGM plates. Abrate [125] used the Navier method based on CPT to investigate the natural vibration behavior of FGM plates with properly chosen reference planes that can be idealized as homogeneous plates. The effect of bending-stretching coupling can be ignored for selecting an accurate reference surface. Chakraverty and Pradhan [126] used the Rayleigh-Ritz method based on CPT to examine the natural vibration behavior of the FGM plate with different aspect ratios and boundary conditions. They continued their work to the considered thermal effect with exponential material property variations on thin FGM plates [42]. Matsunaga [127, 128] applied Navier's method based on 2D higher shear deformation theory with considered shear deformation effect to analyze the natural vibration and stability behavior of FGM rectangular plates. Rung and Wang [129] used a differential quadrature method based on CPT to analyze the stability and transverse vibration of the skew thin FGM plate. Fares et al. [130] implemented a refined single layer theory with a mixed variation approach to estimate FGM plates' free vibration and bedding characterization. In the applied theory, the effect of shear correction factor does not require for accounts the both normal strain and transverse shear impact, which is complete consistency with applied boundary conditions at the top and bottom edge of the FGM plate. Cinefra et al. [131] used Carrera's Unified formulation (CUF) based on extended RMVT for the natural vibration response of multi-layered FGM shell combination with different shell theory by Navier's method. For better accuracy, classical theories were validated with layer-wise, mixed shell, and higher-order theories. Hosseini-Hashemi et al. [132] used the Navier solution based on Mindlin's FSDT to present the exact form of frequency equation for analyzing the free vibration behavior of rectangular FGM plate. Hosseini-Hashemi et al. [133] implemented a Levi-type method to study the free vibration response rectangular FGM plate based on first order deformation theory, where the two contrary edges are simply supported and others under different combinations of boundary conditions. For continued his work, Hosseini-Hashemi et al. [134] applied the same principle method for analyzing the free vibration response of the FGM plate based on Reddy's TSDT. Xiang et al. [135] used Euler-Lagrange method based on Reddy TSDT to investigate the natural vibration for rectangular FGM and composite plates. Benachour et al. [136] implemented the Ritz method and Navier's method along with four variable refine plate theory to study the free vibration behavior of rectangular FGM plates. The parabolic distribution of the transverse shear strain and shear effect along the transverse direction is considered with applied theory and not required of the shear correction factor, unlike FSDT. Hadji et al. [137] applied the method in conjunction with Shimpi's four variable refine plate theory to study the free vibration characterization of the FGM sandwich plate. The obtained results were compared with the reported results of 3D elasticity solution, CPT, FSDT, TSDT, and SSDT. It was noticed that the proposed results predict more accurate results. Alijani et al. [138] applied multiple scales and the Galerkin method based on Donnell's nonlinear shallow shell theory for the free and forced vibration analysis of the FGM doubly curved shallow shells. Li et al. [139] implemented CPT with the physical neutral surface to examine the solution of bending, buckling, and natural vibration characterization of isotropic thin FGM plates and those of corresponding homogeneous isotropic plates. Uymaz et al. [140] applied the Chebyshev polynomials displacement function and the Ritz method along with HSDT for analyzing the free vibration behavior of the FGM plate. Zhang [141] implemented the Ritz method based on Reddy's TSDT and physical neutral surface to examine the free vibration, nonlinear post-buckling, and bending response of the FGM plate. Thai et al. [142] used the quadratic variation method in conjunction with efficient shear deformation theory to investigate the free vibration response of the FGM plates. Ungbhakorn and Wattanasakulpong [143] applied TSDT with distributed patch mass method to examine FGM plates' free and thermo elastic vibration response. The uniform, linear and nonlinear temperature distribution was considered through the FGM plate's thickness direction. The notable effects of size, magnitude, and location of the patch mass on vibrational frequency were reported. Su et al. [144] applied a unified solution method based on FSDT and modified Fourier series, including an exact solution to identify free vibration characterization of FGM cylindrical, conical shells, and annular plates. Sundararajan et al. [145] used FSDT with eight nodded shear flexible quadrilateral method based on a continuous approach to investigate the large amplitude free flexural vibration response of FGM plates. Pradyumna and Bandopadhyay [146] implement eight nodded C<sup>o</sup>

continuity methods in conjunction with HSDT to analyze the free vibration response of simply supported rectangular FGM curved panels. Alijani et al. [147] used the finite element method (FEM) and FSDT to analyze the non-linear free vibration of the FGM doubly curved shallow shell. Talha and Singh [148,149] applied a C<sup>0</sup> continuous method based on 13 degrees of freedom (dof) at each node with higher shear deformation theory for investigating the free and large amplitude flexural vibration response of FGM plates. Malekzadeh and Shojaee [150] used Newmark's time integration method in conjunction with eight nodded solid element with FSDT to study the free vibration response of FGM plates under different boundary conditions subjected to moving heat source. Yang and Shen [151] implemented modal superposition and Galerkin approach based on Reddy's HSDT for analyzing the free and forced vibration behavior of initially stressed FGM plate under thermal environment conditions. Later, Kitipornchai et al. [152, 153] applied a similar mathematical methodology to investigate the random free vibration and initial imperfection of non-linear vibration of laminated FGM plate working under thermal environment conditions. Qian et al. [154] employed the meshless Petrov-Galerkin (MLPG) method with HSDT plate theory to analyze FGM plates' free and forced vibration behavior. Here, the transverse normal and shear deformation were both considered. The obtained frequency results were also validated with the analytical method and Galerkin FEM. Dai et al. [155] applied the Meshfree radial point interpolation method based on FSDT to examine FGM plates' static and dynamic behavior. Chen [156] employed Galerkin's and the Runge-Kutta methods in conjunction with von-Karman assumptions to analyze the free vibration response of initially stressed FGM plates. The proposed method was applied to convert partial GDE into ordinary differential equations (ODE) and explained by the Runge-Kutta method. Ferreira et al. [157] applied the global collocation method and multi quadratic radial function based on FSDT and TSDT to present the solution of free vibration characterization of FGM plates. Fung and Chen [158] applied the Runge-Kutta, Galerkin, and perturbation techniques with FSDT based on geometric imperfection and von Kamen's non-linearity to study the non-linear vibration response of initially stressed FGM rectangular plates. Roque et al. [159] employed a multi quadratic radial basis function with TSDT to investigate the free vibration response of FGM plates under arbitrary boundary conditions. Zhao et al. [160] applied the kp-Ritz method based on FSDT to analyze the natural vibration response of the FGM cylindrical shell. Further, Zhao et al. [161] implemented the same method with FSDT to analyze the static behaviour and natural vibration response of FGM plates with thermo mechanical load. The obtained results

determined that the various numerical values of shear correction factors had a less significant effect on frequency while side to thickness was generally  $\geq 10$ . Malekzadeh and Beni [162] employed the differential quadrature method, a geometric mapping technique, and FSDT to examine the free vibration response of thermal environment-based FGM rectangular and skew plates. Nie and Zhong [163] implemented the state space method in conjunction with the DQM to study the dynamic behavior of multi-directional material property of FGM plate where the material property exponentially varied through-thickness and radial both the directions. Zahedinejad et al. [164] applied a differential quadrature method based on trigonometric functions to analyze FGM curved panels' three-dimensional (3D) free vibration response. Further, Malekzadeh et al. [165] employed the same differential quadrature method with 3D elasticity theory to study the free vibration characterization under thermal environment conditions of FGM thick annular plates. Yas and Aragh [166] used an elasticity solution with a generalized quadrature method to analyze the free vibration response of four parametric FGM fiber-orientated cylindrical panels. Wu et al. [167] applied the Galerkin and collocation method based on differential reproducing kernel interpolation to investigate the threedimensional (3D) natural vibration response of FGM sandwich plates. Zhu and Liew [168] used the Petrov-Galerkin method and FSDT to develop an exact shape function with the Kronecker delta function property to analyze the free vibration response of FGM rectangular plates. Alijani et al. [169] used multi model energy approach and pseudo arclength continuation method based on collocation technique with two different higher-order theories to study the impact of thermal load on nonlinear vibration of FGM doubly curved shells. Neves et al. [170] employed the radial basis function collocation method with Carrera's Unified formulation (CUF) in collaboration with higher deformation theory to investigate the free vibration behavior of FGM shells. Zhu and Liew [171] formulate the meshless method by the Kriging interpolation solution with FSDT and von-Karman non-linearity technique for investigating the free vibration behavior of rectangular FGM plates. Reddy and Cheng [172] applied a 3D asymptotic approach in terms of transfer matrix with the Mori-Tanaka method for investigating the harmonic vibration response of rectangular FGM plates. Vel and Batra [173] implemented the power series expansion method together with FSDT to examine the free and forced vibration problem of simplysupported rectangular FGM plates. The obtained natural frequency results were validated with reported results of TSDT, FSDT, and CPT. It was noticed that the FSDT frequency results perform better than TSDT reported results. Nie and Zhong [174] employed the 1D quadrature differential method with the state space method and a semi-analytical approach for 3D analyses of the free vibration problem of FDM circular plates under different boundary

conditions. It was noticed that the semi-analytical technique required less computational time, and it has a high advantage in the obtained computational efficiency. Further, Nie and Zhong [175] extended the same methodology for analyses of the free and forced vibration characterization of simply supported FGM annular plates. Dong [176] used the Chebyshev-Ritz method with 3D elasticity theory to investigate the free vibration response of FGM circular plates under different boundary conditions. The material property variations were described using exponential law and power law function. Tsai and Wu [177] employed the asymptotic expansion method and 3D elasticity theory for estimating the free vibration behavior of simply-supported FGM magneto electro elastic double-curved shell. It was observed that the influence of material property significantly impacts the electric and magnetic fields. Li et al. [178] applied the Ritz energy method and Chebyshev polynomial technique for analyzing the 3D free vibration response of simply supported rectangular FGM plates under thermal environment conditions subjected to uniform and non-uniform temperature rise through the thickness direction. Hashemi et al. [179] demonstrated an exact closed-form solution using Levinson's method and 3D elasticity theory to examine the inplane and out-of-plane free vibration response of the FGM plate. Recently, Ali et al. [180] used Kirchhoff's-love plate theory to formulate a dynamic stiffness matrix for free vibration analysis to investigate vibration frequencies and mode shapes of the porous FGM plate.

## 2.2.2 Literature review based on free vibration analysis of FGM plate structures with elastic foundations

The dynamic response of rectangular plate structural components resting on Winkler and Pasternak elastic foundations produces an important significant characterization in the structure design. The characterization response of these structures is generally affected by the interaction of different design parameters between the plate and elastic foundations. In this context, two simplest models are implemented to explain the interaction between the plate surface and elastic foundation. The simplest is the one-parameter mathematical model, the Winkler model [43], where normal pressure is used to estimate the mechanical response of the plate resting on the elastic foundation. The Winkler foundation is considered equivalent to continuously assigned linear springs without any coupling effects between them. Later, Pasternak [44] improvised this model and proposed two dependent parameters (normal pressure and shear layer) to account for the shear interaction in the spring system. The

existence of shear interaction among the spring elements is accomplished by connecting the ends of the springs to the plate that only undergoes transverse shear deformation [47, 48]. Winkler and Pasternak models are widely used to model the foundations of railroad tracks, airports, motorways, concrete slabs, rigid concrete pavement for highways, and raft foundations [44, 47,48].

The existing literature based on the research work on plates resting on elastic foundations is very rich. Apart from this, the available research work related to the vibration response of rectangular FGM plates supported on elastic foundation parameters have high potential values because of the characteristic advantage of functionally graded material over traditional materials. So, this present research works are described literature review based on vibration characterization of the FGM plates resting on elastic foundations. For example, Yang and Shen [181] applied the one-dimensional differential quadrature method (DQM) and Galerkin superposition modal along with CPT to study the natural vibration and mode shapes of initially stressed FGM plates resting on elastic foundations. Lam et al. [182] implemented Kirchhoff's plate theory (or classical plate theory) with an exact canonical solution to investigate the buckling and bending vibration response of the Levy-type rectangular isotropic plates embedded on elastic foundations. Han et al. [183] used the finite element method with FSDT to extract the natural vibration frequency and buckling response with considered transverse shear strain and rotatory inertia of the simple-supported S-FGM plate resting on elastic foundations. Amini et al. [184] applied Ritz and Chebyshev polynomial method to carry out the natural frequency of rectangular clamped and simple supported FGM plate resting on Winkler's elastic foundation. Lu et al. [185] applied the space state method to examine the natural vibration behavior of exponentially FGM plates supported on Pasternak elastic foundation. Atmane et al. [186] used the Navier method to compute the eigenvalues based on the higher deformation theory of the simply supported P-FGM and E-FGM plate resting on the Winkler and Pasternak elastic foundation. Baferani et al. [187] reported the free vibration response of rectangular FGM plates embedded on the elastic medium using TSDT. They used an analytical method to decouple the differential equation of motion. These decoupled equations based on the elastic foundation of the plate were solved using levy type solution. Omurtaz et al. [188] applied the Gateaux differential method based on Kirchhoff's plate theory to employ the free vibration response of a rectangular plate with Pasternak elastic foundation. Farid et al. [189] used the differential quadrature method based on 3D elasticity theory to analyze the free vibration behavior of an initially stressed thick simply-supported

FGM panel embedded on a two-parameter elastic foundation under applied thermal environment conditions. Malekzadeh et al. [190] employed a semi-analytical method in conjunction with differential quadrature method and 3D theory of elasticity for analyzing the free vibration problems of FGM plate resting on two-parameters elastic foundation under applied two different edges were simply supported boundary conditions. Huang et al. [191] applied the state space method along with the three-dimension elasticity theory to compute the natural frequency of the FGM plate supported on the Winkler and Pasternak elastic foundation. The material properties vary exponentially through the transverse direction of the plate. It is obtained from the results that the elastic foundation influences a significant impact on the mechanical behavior of FGM plates. Hosseini-Hashemi et al. [192] implemented a differential quadrature method with CPT for examining the free vibration problems of FGM annular sector plates embedded on Pasternak elastic foundation and subjected to in-plane compressive load under applied simply supported and clamped boundary conditions. Benyocef et al. [193] applied a new hyperbolic displacement method along with subjected to a sinusoidally distributed load (SL) or a transverse uniform load (UL) to analyze the free vibration and static behavior of simply-supported FGM plates resting on two-parameters elastic foundation. Hosseini-Hashemi et al. [194] used the differential quadrature method (DQM) based on CPT to examine the natural vibration behavior of annular sector thin plates supported on elastic foundation. Tajeddini et al. [195] employed the polynomial-Ritz method in conjunction with exact elasticity theory for analysing the natural frequency of the annular FGM pates supported on the Pasternak elastic foundation. Rad et al. [196] used a semianalytical approach and differential quadrature method based on the 3D theory of elasticity to compute the natural vibration response of an annular FGM plate resting on Winkler and Pasternak elastic foundation. Shariyat et al. [197] applied a differential transformation and a semi-analytical method to extract the free vibration problems of two-direction FGM circular plates embedded on elastic foundation. Shen et al. [198] implemented the von Karman type method based on HSDT for analyzing the free vibration, and bending problems of simplysupported FGM plates supported Pasternak type elastic foundation subjected to sinusoidal load combined with compressive load at the edge of the plate. Hosseini-Hashemi et al. [199] employed the Ritz method based on a set of static Timoshenko beam function and Mindlin plate theory to compute the free elastic vibration with bending problems of rectangular FGM plates subjected to in-plane load under different boundary conditions. Xiang [200] used the state space method based on Mindlin plate theory to estimate the natural vibration response of FGM rectangular plate supported on a non-homogenous Winkler type elastic foundation with Levy type boundary condition. Mohammadi et al. [201] employed an analytical method in conjunction with Kirchhoff's plate theory and Levy-type solution for investigating the natural vibration and bending behavior of rectangular FGM plate supported on Winkler and Pasternak elastic foundation. Alibeigloo [202] applied the state space method based on the Fourier series to estimate the free vibration response of the FGM plate supported on an elastic foundation subjected to a transverse load. Zhang et al. [203] employed the Navier method along with strain gradient elasticity theory and refined shear deformation theory for investigating the buckling, bending, and natural vibration response of simply-supported micro FGM plates embedded on Winkler-Pasternak elastic foundation. Asanjarani et al. [204] applied the differential quadrature method based on FSDT to analyze the natural vibration behavior of rectangular FGM conical shells supported on the Winkler and Pasternak elastic foundation. Tornabene et al. [205] employed a generalized differential quadrature method along with FSDT for estimating the natural vibration response of laminated double-curved shells supported on the Winkler and Pasternak elastic foundation. Bahadori et al. [206] applied the Navier-differential quadrature solution method and FSDT for analyzing the free vibration dynamic response of FGM cylindrical shells supported on Winkler and Pasternak elastic foundation. Baferani et al. [207] employed analytical method and Kirchhoff's plate theory to analyze the free vibration response of simply supported thin annular sector FGM plates embedded on Winkler and Pasternak elastic foundation. Akavci et al. [208] used Navier's solution technique based on hyperbolic HSDT to analyze the natural vibration response of the FGM plate with a two-parameter (Winkler-Pasternak) elastic foundation. Thai et al. [209] applied the Navier method with refined plate theory to estimate the out-of-plate free vibration response of the FGM plate with a two-parameter Pasternak elastic foundation. Further continuing previous work, Thai et al. [210] used the Navier method under refined plate theory to estimate in-plane free vibration of the FGM plate resting on the Pasternak foundation. Fallah et al. [211] investigated the natural frequency of moderately thick FGM plates embedded on the Winkler model. They extended the Kantoroich method and infinite power series to develop the semi-analytical solution to determine the natural frequency of the plate. Han et al. [212] implemented Bolotin's method with four variable refine plate theory to analyze the dynamic stability characterization based on the physical neutral surface of S-FGM plate resting on elastic foundation under different boundary conditions. Sheikholeslami et al. [213] implemented an analytical method based on normal deformable and HSDT to

estimate the free vibration analysis of FGM rectangular plates with a two-parameter elastic foundation. Tajeddini et al. [214] employed the polynomial Ritz-method based on linear, small strain exact elasticity theory for examining the natural frequency of annular isotropic and FGM plates resting on Pasternak elastic foundation. Yas et al. [215] applied the differential quadrature method in conjunction with the 3D theory of elasticity to compute the natural vibration response of simply-clamped, free-clamped, and clamped-clamped FGM plates resting on Pasternak or two-parameter elastic model. Sobhy et al. [216] used an analytical method based on sinusoidal deformation theory to examine the buckling and free vibration response of exponentially FGM sandwich plate embedded on Pasternak elastic foundation. Mantari et al. [217] applied the Navier-type method along with non-polynomial HSDT to study the free vibration analyses of FGM plate supported on Pasternak elastic foundation. Milad et al. [218] applied a numerical differential quadrature method based on FSDT, analyzing the natural vibration behavior of FGM annular sectorial plates resting on the Winkler-Pasternak elastic foundation. Dehghan et al. [219] used differential quadrature and finite element method three-dimension elasticity theory to study the free vibration and buckling behavior of rectangular FGM plates surrounding the Pasternak elastic foundation. Kamarian et al. [220] employed the differential quadrature method in conjunction with threedimensional elasticity theory for examining the natural frequency of the FGM plate resting on the Pasternak elastic foundation. Kiani et al. [221] implemented the Navier type method based on FSDT and the modified Sanders shell theory to study the free vibration and dynamic response of FGM doubly curved panels surrounding the Pasternak type elastic foundation. Dehghany et al. [222] used the Navier solution and three-dimensional elasticity theory to examine the free vibration analysis of a simply supported rectangular FGM plate embedded on two-parameter elastic model. Ameur et al. [223] employed the Navier solution based on a new trigonometric shear deformation plate theory to investigate FGM plates' free vibration response with elastic foundations. Bahmyari et al. [224] implemented Galerkin Method (EFGM) along with FSDT to carry out the free vibration behavior of the FGM plate with point supports resting on a two-parameter elastic foundation. Huang et al. [225] applied a Half Boundary Method (HBM) based on the Green function to examine the free vibration characterization of the FGM plate resting on the Winkler elastic foundation. Bakhtiari-Nejad et al. [226] used the Rayleigh-Ritz method based on the Sanders-Koiter shell theory to analyze the natural frequency of the FGM cylindrical shells supported Winkler and Pasternak elastic foundation. Mantari et al. [227] applied the Navier type closed-form solution in conjunction with quasi-3D hybrid type hyper shear deformation theory to examine the natural vibration response of FGM plate embedded on elastic foundation. Shahsavari et al. [228] Galerkin method based on a novel quasi-3D hyperbolic theory to compute the natural frequency of FGM plate resting Winkler-Pasternak elastic foundation. Bouafia et al. [229] applied Navier's technique and higher-order quasi-3D formulation to study the free vibration and bending analysis of FGM plate supported on elastic foundation. Kumar et al. [230] employed Galerkin-Vlasov's method with FSDT for analyzing the free vibration behavior of FGM porous plate supported on Winkler and Pasternak elastic foundation. Peng et al. [231] applied the moving Kriging (MK) approximation method and FSDT for examining the free and static vibration response of the FGM plate embedded on elastic foundation. Benferhat et al. [232] implemented the Navier method considering a neutral surface to analyze the free vibration response with a simply supported FG plate resting on the Winkler-Pasternak foundation. They considered the shear deformation effect by implementing the refined shear deformation theory and developed exact shearing strain, and no shear correction factor is further needed for free vibration analysis of FG plate. Shen et al. [233] used the Rayleigh-Ritz method with Reissner-Mindlin plate theory to study the free and forced vibration of four free edges plates resting on a Pasternak elastic foundation. The modal superposition procedure is considered with the Mindlin-Goodman approach to estimate the dynamic behavior of the free edge plate. Omurtag et al. [234] applied Kirchhoff plate theory with FEM to analyze the natural vibration of the plate resting on the Winkler and Pasternak foundation. Radakovic et al. [235] used the von Karman method with HSDT to study the natural vibration response of the FGM plate supported on the Winkler and Pasternak elastic foundation. Guellil et al. [236] used the Navier solution method based on HSDT for analyzing the free vibration and bending response of the FGM plate resting on Pasternak elastic foundations. Abdelbari et al. [237] implemented the Navier-type analytical method based on a simple hyperbolic shear deformation theory to examine the natural vibration behavior of the FGM plate supported on elastic foundation. Zaoui et al. [238] employed Navier's method and a two-dimensional (2D) and quasi three-dimensional (3D) deformation theory to carry out the free vibration response of FGM plate with elastic foundation. Singh et al. [239] employed the Rayleigh-Ritz method and CPT to study the out-of-plane free vibration behavior of FGM plate with Winkler-Pasternak elastic foundation under different boundary conditions. Saidi et al. [240] applied the Navier-type method and four unknown shear deformation theory to investigate the natural vibration behavior of simply-supported FGM rectangular plates supported on Winkler and Pasternak elastic foundation. Nebab et al. [241] used a Navier solution based on four unknown HSDT to examine the natural vibration behavior of an FGM rectangular plate supported on a two-parameter elastic foundation. Tran et al. [242] applied the Galerkin method with FSDT based on the Lekhnitsky stiffener technique to study the natural vibration characterization of FGM plate resting on elastic foundation under different boundary conditions. Besseghier et al. [243] used the Navier-type method and a novel nonlocal refined trigonometric shear deformation theory with refined four-variable shear deformation for analyzing the natural vibration response of size-dependent nanoscale FGM plate embedded on elastic foundation. Benahmed et al. [244] implemented Navier's analytical method based on simple quasi-3D hyperbolic deformation theory to extract the natural vibration and the bending response of an FGM rectangular plate resting on a two-parameter elastic foundation. Huang et al. [245] implemented a finite strip method to estimate the static and free vibration behavior of a mixed boundary condition plate resting on the Winkler elastic foundation. Continuing previous work, Huang et al. [246] used the state space method considered with the three-dimensional (3D) theory of elasticity to estimate the natural vibration behavior FGM plate embedded on the Winkler and Pasternak elastic foundation. Farid et al. [247] used the quadrature method in conjunction with trigonometric function in the transverse and longitudinal direction to carry out the free vibration response of the initially stressed FGM plate supported on the Winkler and Pasternak foundation. Kamarian et al. [248] implemented the generalized differential quadrature method (GDQM) considered with a three-dimensional theory of elasticity to characterize the natural vibration behavior of the three-parameter FG plate structure resting on the Pasternak foundation. Akavci et al. [249] applied a hyperbolic shear deformation approach with the Navier method to carry out the natural frequency of the FG plate resting on the Pasternak foundation. Malekzadeh et al. [250] applied DQM along with FSDT to analyze the free vibration response of a plate supported on an elastic foundation. Continuing his work, Malekzadeh et al. [251] applied the differential quadrature method (DQM) with a three-dimensional theory of elasticity to carry out the free vibration behavior of FGM plates with the Pasternak foundation. Su et al. [252] used the FSDT considered with the Fourier-Ritz solution method to carry out the free frequency of the laminated FGM plate with a laminated FGM plate resting on an elastic foundation. Mesksi [253] investigated the free vibration behavior of FG plates supported on elastic foundations. They used the Navier method with novel FSDT to extract the natural frequency of the FGM plate. Various authors and co-authors, Ramu et al. [254], Ozgan et al. [255], Ansari t al. [256], Benahmed et al. [257], Jung et al. [258, 259], Gupta et al. [260, 261], Vinh Tran et al. [262], Pridha et al. [263, 264] applied different analytical technique with different plate theories to carried out the natural frequency of the FGM plates resting on a different combination on elastic foundation with different boundary conditions.

## **CHAPTER 3**

# Free Vibration Analysis of P-FGM Plate Resting on Elastic Foundation

#### 3.1 Introduction

This present chapter of the thesis explained the development of dynamic stiffness method to analyse the free vibration response of thin rectangular P-FGM plates supported on Winkler-Pasternak elastic foundations. As described earlier in Chapter 1, the material property variation of functionally graded material (P-FGM) plate is defined by power-law function where properties are varying continuously through the transverse direction in terms of volume fraction of the plate. A well-known CPT, along with Hamilton's principle, is implemented to develop the general governing partial differential motion equation, and Levytype boundary condition is applied on FGM plate supported by Winkler and Pasternak elastic foundation. The Wittrick and William (W-W) algorithm is enforced as a solution technique to carry out the desired natural frequencies and mode shape of the global DS matrix of the FGM plate. The obtained natural frequencies for isotropic, power-law functionally graded plate with and without foundation supported on Winkler and Pasternak elastic foundation are validated with published literature. The present work also reported the effect of the FGM plate's geometric and material parameters (material gradient index, Young's Modulus and mass density ratio, aspect ratio, edges condition). A new set of DSM eigenvalues results for a functionally graded plate with elastic foundation are presented.

The description of the remaining part of the present work can be described as follows. The main contribution of the present work is explained in section 3.1. The theoretical formulation of the P-FGM plate, (material property variation and modelling of elastic foundation) is reported in section 3.2. The mathematical formulation of DSM and free vibration governing equation are developed using Hamilton's principle. In section 3.3, new sets of DSM results are extracted and the effect of foundation geometric and material parameters on natural frequency is reported. The conclusion is explained in the last section, 3.4.

### 3.2. Theoretical formulation

### 3.2.1 Geometrical configuration of the P-FGM plate

The Cartesian coordinate parameters of a thin rectangular P-FGM plate are described in Fig. 3.1, where the geometric descriptions of the plate are as length(a), width (b), and plate thickness (h). The plate is formed by the combination of ceramic and metal constituents, where the upper part of the given plate is highly rich in ceramic, and the lower part is highly rich in metallic. Here, the material property varies along the direction of plate thickness by power law [37, 39] in the form of volume fraction as defined by Eq. (3.1).

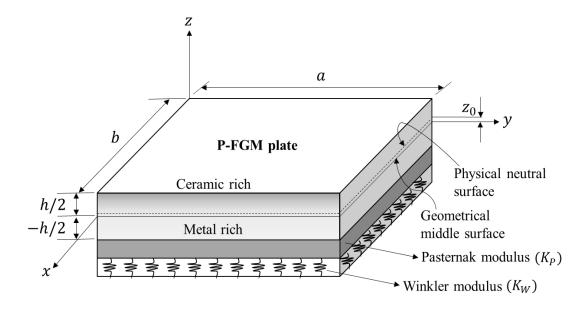


Fig. 3.1: Schematic representation of thin rectangular power-law FGM plate on elastic foundation

The expression of volume fraction of power-law function can be given by

$$V_c(z) = \frac{1}{2} \left( 1 + \frac{2z}{h} \right)^p$$
 and  $V_m(z) = 1 - V_c(z)$ ; where  $0 \le p \le \infty$  (3.1)

where (p) represents the non-negative material parameter and is called material gradient index, which describes the volume fraction of given material. The application of mixture rule is applied to determine the material property and is given by Eq. (3.2).

$$E(z) = V_c(z)E_c + V_m(z)E_m$$

$$= (E_c - E_m) \frac{1}{2} \left( 1 + \frac{2z}{h} \right)^p + E_m, \tag{3.2}$$

where the material property of metal and ceramic are represented by  $E_m$  and  $E_c$ , respectively. The material property variation of the power-law FGM plate is indicated by density  $\rho(z)$  and Young's modulus E(z), respectively, and represent by Eq. (3.2).

Table 3.1: [101-103] represents the common values of metal and ceramic material property and Aluminium (Al) and Alumina ( $Al_2O_3$ ) as metal and ceramic constituents, respectively.

	Material properties of FGM		
Material	Elasticity	Mass	Poisson's
	modulus (GPa)	density (kg/m <sup>3</sup> )	ratio
(Alumina): Al <sub>2</sub> O <sub>3</sub>	$E_{c} = 380$	$\rho_{\rm c} = 3800$	v = 0.3
(Aluminium): Al	$E_m = 70$	$ \rho_m = 2707 $	v = 0.3

Section 1.2.1 of Chapter 1 described the variation of material property (Young's modulus E(z) and density  $\rho(z)$ ) of the P-FGM plate, where at material gradient index (p=0), the effect of given material property does not change in the thickness direction of the P-FGM plate, and it's considered a homogenous isotropic plate in that direction. At applying the large value of material gradient index (p), by nature, the P-FGM plate act as a bi-material where the plate top side is ceramic rich, and the bottom part is metal rich, respectively, as determined in Fig.1.4 (a).

### 3.2.2 Elastic foundation models

A Winkler model [43] is defined as a closed vertical linear spring system that is mutually independent of each other, and the stiffness of the spring is considered equivalent to the elastic modulus of the medium. Winkler model has been improved by Pasternak [44], where the shear interaction is applied between the spring system and the whole geometrical configuration is obtained by connecting the springs ends to the P-FGM plate that only undergoes transverse shear deformation as shown in Fig. 3.1. The main dominance advantage of the Pasternak model is that it applied both normal pressure and transverse shear deformation at a given system of the surrounding elastic medium. By taking this advantage, the Pasternak model [45-47] is widely used for analyzing the structure interaction response of free vibration supported on an elastic foundation.

The normal force [48] of the elastic foundation can be expressed as

 $q_{\text{Pasternak}} = k_W w - k_P \nabla^2 w$ 

$$q_{\mathrm{Winkler}} = k_W w$$

where the normal force of the foundation is represented by q and vertical displacement represented by w,  $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ ,  $k_W$  and  $k_P$  indicate the Winkler and Pasternak elastic modulus, respectively.

(3.3)

### 3.2.3 Description of classical plate theory with physical neutral surface

According to classical plate theory, the displacement component (kinematic variable) of the plate can be given by

$$u_{x} = u_{o}(x, y) - z \frac{\partial w}{\partial x}$$

$$v_{y} = v_{o}(x, y) - z \frac{\partial w}{\partial y}$$

$$w_{z} = w(x, y)$$
(3.4)

The middle-surface in-plane displacement component of the power-law FGM plate can be represented as  $u_o(x,y)$  and  $v_o(x,y)$ , respectively. The material property of the FGM plate is nonhomogenous nature in the transverse direction, and due to this, a significant amount of out-plane displacement is induced, which cannot be neglected [7, 6]. Because of this, the physical neutral surface (PNS) does not coincide with the geometric middle surface, and taking this reason, the bending stretching coupling effect is present for the P-FGM plate, as shown in Eq. (3.4). If the effect of PNS is applied on the P-FGM plate, the coupling phenomena of Eq. (3.4) can be eliminated [101, 102]. The idea of new coordinates system  $z_{ns} = (z - z_0)$  is implemented at the neutral surface of the FGM plate, the distance between the neutral surface to the middle surface of the geometry is represented by  $z_0$ , as shown in Fig.3.1. As a result, the in-plane displacement components of the FGM plate are very small in the new coordinates system and can be neglected [101, 102].

As per the new coordinates system, the displacement component of the FGM plate can be given by

$$u_{x} = -z_{ns} \frac{\partial w}{\partial x} = -(z - z_{0}) \frac{\partial w}{\partial x}$$

$$v_{y} = -z_{ns} \frac{\partial w}{\partial y} = -(z - z_{0}) \frac{\partial w}{\partial y}$$

$$w_{z} = w(x, y)$$
(3.5)

The associated strains constraint of the Eq. (5) can be given by

$$\varepsilon_{xx} = -z_{ns} \frac{\partial^2 w}{\partial x^2}; \quad \varepsilon_{yy} = -z_{ns} \frac{\partial^2 w}{\partial y^2}; \quad \gamma_{xy} = -2z_{ns} \frac{\partial^2 w}{\partial xy}$$
 (3.6)

where normal strains in a particular x and y direction is represented by  $\varepsilon_{xx}$  and  $\varepsilon_{yy}$  respectively, and  $\gamma_{xy}$  denotes the x-y plane shear strain.

The FGM plate material constituents follow the basic principle of Hook's law, and as per this law, the relationship of stress and strain are given in the following matrix and expressed as

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$
(3.7)

The normal stresses of the plate can be represented as  $\sigma_{xx}$  and  $\sigma_{yy}$  in the x and y direction, respectively, and the shear stress of x-y plane is representing by  $\tau_{xy}$ . The plate reduced stiffness components  $(Q_{ij})$  can be expressed as

$$Q_{11} = Q_{22} = \frac{E(z_{ns})}{1 - v^2}; \quad Q_{12} = Q_{21} = \frac{vE(z_{ns})}{1 - v^2} \text{ and } \quad Q_{66} = \frac{E(z_{ns})}{2(1 + v)}$$
 (3.8)

To examine the PNS  $(z_0)$ , the plate total axial force in the particular x, y direction should be zero. Therefore

$$\sum F_{x} = \int_{-h/2-z_{0}}^{h/2-z_{0}} \sigma_{xx} dA = 0$$
 (3.9)

which gives to

$$z_0 = \frac{\int_{-h/2}^{h/2} E(z)zdz}{\int_{-h/2}^{h/2} E(z)dz}$$

$$= \frac{hp (E_c - E_m)}{2(p+2)(E_c + pE_m)} = \frac{hp (E_{rat} - 1)}{2(p+2)(E_{rat} + p)}$$
(3.10)

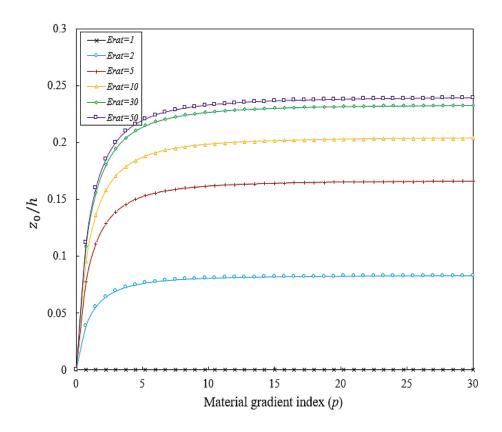


Fig. 3.2: Shifting of nondimensional  $(z_0/h)$  parameter with material gradient index (p)

Eq. (3.10) represents that the PNS ( $z_0$ ) of the FGM plate is the function of Young's modulus ratio ( $E_{rat} = E_c/E_m$ ) and material gradient index (p). The effect of  $E_{rat}$  and p on PNS is shown in Fig. 1.2.1. For a particularly given value of Young's modulus ratio ( $E_{rat} = 1$ ) with  $z_0 = 0$ , the power-law FGM plate is converted to a homogenous isotropic plate, and at this condition, PNS exactly matched (coincides) to the mid surface of the plate. Eq. (3.10) shows that, as Young's modulus ratio ( $E_{rat}$ ) increases, the corresponding  $z_0$  value also increases. It is obtained from Fig.3.2 that as  $E_{rat}$  increases, the PNS is moved away from the middle surface of the FGM plate and shifted as for the highly ceramic side of the plate. This phenomenon is due to the higher stiffness value of ceramic constituents at the top part compared with metal constituents at the bottom part of the power-law FGM plate.

### 3.2.4. The governing free vibration equation of motion of FGM plate

The governing motion equation of the FGM plate for free vibration is carried out by implementing general Hamilton's principle.

As per Hamilton's principle,

$$\delta \int (T - (U + U_{EM})dt = 0 \tag{3.11}$$

where T, U represent the kinetic energy, and strain energy of the plate,  $U_{EM}$  represent the strain energy due to the elastic foundation and can be given by

$$T = \frac{1}{2} \int_{V} \rho \left( \left( \frac{\partial u_{x}}{\partial t} \right)^{2} + \left( \frac{\partial v_{y}}{\partial t} \right)^{2} + \left( \frac{\partial w_{z}}{\partial t} \right)^{2} \right) dV$$
(3.12)

The strain energy (U) of the P-FGM plate is expressed as

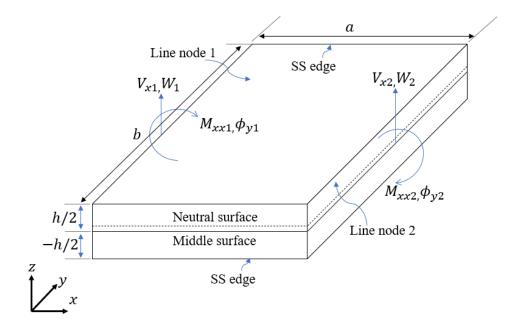
$$U = \frac{1}{2} \int_{V} \left( \sigma_{xx} \varepsilon_{xx} + \sigma_{yy} \varepsilon_{yy} + \tau_{xy} \gamma_{xy} \right) dV \tag{3.13}$$

The power-law functionally graded plate is supported on an elastic foundation, therefore the foundation strain energy  $(U_{EM})$  is expressed as [104].

$$U_{EM} = \int_{\Omega} \left[ k_W w \delta w + k_P \left( \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial y} \right) \right] dx dy$$
(3.14)

Eq.(3.5), the displacement components are substituted into Eqs. (3.11-3.14) and the relationship of stress-strain of Eqs. (3.6-3.7) is substituted into Eq. (3.13) with applied Hamilton's principle, the equation of motion for free transverse direction vibration of P-FGM plate and general boundary condition can be formulated as given by Eq. (3.15) and (3.16) respectively.

$$D_{FGM}\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + I_0\frac{\partial^2 w}{\partial t^2} + k_W\frac{\partial^2 w}{\partial t^2} + k_P\left(\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\partial^4 w}{\partial y^2 \partial t^2}\right) = 0$$
(3.15)



**Fig.3.3:** Schematic representation of force and displacements boundary conditions for a plate element.

The plate element natural boundary conditions are given as present in Fig. (3.3).

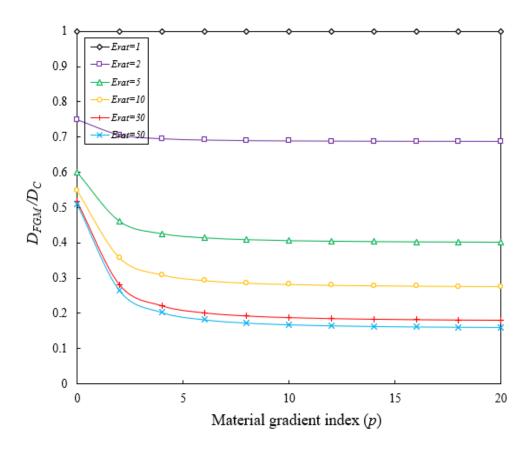
$$V_{x} = \left[ -D_{FGM} \left( \frac{\partial^{3} w}{\partial x^{3}} + (2 - v) \frac{\partial^{3} w}{\partial x \partial y^{2}} \right) \right] \delta w$$

$$M_{xx} = -D_{FGM} \left( \frac{\partial^{2} w}{\partial x^{2}} + v \frac{\partial^{2} w}{\partial y^{2}} \right) \delta \phi_{y}$$
(3.16)

where flexure rigidity of the P-FGM plate is represented by  $D_{FGM}$ . The  $V_x$ , and  $M_{xx}$  represent the shear force and bending moment of the plate, respectively. The analytical mathematical expression of these parameters can be expressed in Appendix A. In this study, nondimensional parameters  $D_{FGM}/D_C$  are applied where the value of  $D_C = E_c h^3/12(1 - \nu^2)$ .

For a particular value of Young's modulus ratio  $E_{rat} = 1$ , the power-law functionally graded plate is developed as a highly ceramic-rich plate with  $D_{FGM}/D_C = 1$ , and the corresponding P-FGM plate becomes a homogenous isotropic plate. Fig.5 represents the effect of  $D_{FGM}/D_C$  for six different values of Young's modulus ratio ( $E_{rat}$ ) with different values of material gradient index. Fig.5 shows that the parametric value of  $D_{FGM}/D_C$  decreases as Young's modulus ratio increases with the material gradient index, and the metallic constituents of the

FG plate are increases, which has less value of Young's modulus and flexure stiffness than that of ceramic constituents.



**Fig. 3.4:** The nondimensional parameter  $D_{FGM}/D_C$  with different p and  $E_{rat}$ .

Eq. (3.16) shows that the plate element natural boundary condition represents that the shear force  $(V_x)$  of the plate element is related to displacement component (w) and the bending moment is related to the rotation  $\phi_y = \frac{\partial w}{\partial x}$  of the plate. The above Eq. (3.15) and Eq. (3.16) are considered key elements for developing the DS matrix, and the methodology to develop the DS matrix is summarized below.

### 3.2.5. Implementation of Levy-type solution

The governing differential equation of Eq. (3.15) is solved by applying the displacement-force boundary conditions. A well-known Levy-type solution is applied in this present work where a simply-supported boundary condition is applied at the two different edges of the plate, and the other remaining edges are considered simply supported, free, and clamped. Eq. (3.15) is completely solved by Levi solution [11], which satisfied the natural boundary condition obtained in Eq. (3.16) and can be expressed as

$$w(x,y,t) = \sum_{m=1}^{\infty} W_m(x)e^{i\omega t}\sin(\alpha_m y) \text{ with } \alpha_m = m\pi/a; (m = 1,2,\dots,\infty)$$
(3.17)

where  $\omega$  represents the unknown frequency.

The obtained Eq. (3.17) is substituted into the Eq. (3.15), a generalized fourth order differential equation is obtained as

$$\frac{d^4 W_m}{dx^4} - \left(2\alpha_m^2 + \frac{k_P \omega^2}{D_{FGM}}\right) \frac{d^2 W_m}{dx^2} + \left(\alpha_m^4 - \frac{k_W \omega^2}{D_{FGM}} - \frac{k_P \omega^2 \alpha_m^2}{D_{FGM}}\right) W_m = 0$$
(3.18)

The developed Eq. (3.18) produce standard four roots, and based on its nature depends, there are only two feasible solutions obtained and given by:

Case 1. 
$$\left(\alpha_m^2 - \frac{k_P \omega^2}{2D_{FGM}}\right) \ge \sqrt{\left(\frac{k_P \omega^2}{2D_{FGM}}\right)^2 + \frac{k_W \omega^2}{D_{FGM}}}$$

In case 1, all roots are real  $(r_{1m}, -r_{1m}, r_{2m}, -r_{2m})$  and the expression are given as

$$r_{1m} = \sqrt{\left(\alpha_m^2 - \frac{k_P \omega^2}{2D_{FGM}}\right) + \sqrt{\left(\frac{k_P \omega^2}{2D_{FGM}}\right)^2 + \frac{k_W \omega^2}{D_{FGM}}}}$$

$$r_{2m} = \sqrt{\left(\alpha_m^2 - \frac{k_P \omega^2}{2D_{FGM}}\right) - \sqrt{\left(\frac{k_P \omega^2}{2D_{FGM}}\right)^2 + \frac{k_W \omega^2}{D_{FGM}}}},$$
(3.19)

The solution is

$$W_m(x) = A_m \cosh(r_{1m}x) + B_m \sinh(r_{1m}x) + C_m \cosh(r_{2m}x) + D_m \sinh(r_{2m}x)$$
(3.20)

Case 2. 
$$\left(\alpha_m^2 - \frac{k_P \omega^2}{2D_{FGM}}\right) < \sqrt{\left(\frac{k_P \omega^2}{2D_{FGM}}\right)^2 + \frac{k_W \omega^2}{D_{FGM}}}$$

In case 2, two roots are real, and two roots are imaginary  $(r_{1m}, -r_{1m}, r_{2m}, -r_{2m})$  and the expression are given as

$$r_{1m} = \sqrt{\left(\alpha_m^2 - \frac{k_P \omega^2}{2D_{FGM}}\right) + \sqrt{\left(\frac{k_P \omega^2}{2D_{FGM}}\right)^2 + \frac{k_W \omega^2}{D_{FGM}}}}$$

$$r_{2m} = \sqrt{-\left(\alpha_m^2 - \frac{k_P \omega^2}{2D_{FGM}}\right) + \sqrt{\left(\frac{k_P \omega^2}{2D_{FGM}}\right)^2 + \frac{k_W \omega^2}{D_{FGM}}}},$$
(3.21)

The solution is:

$$W_m(x) = A_m \cosh(r_{1m}x) + B_m \sinh(r_{1m}x) + C_m \cos(r_{2m}x) + D_m \sin(r_{2m}x)$$
(3.22)

### 3.2.6. Development of DS matrix for P-FGM plate

In the case 1, the development of the dynamic stiffness (DS) matrix is described below. A same pattern is to be used for case 2 but is not described here for brevity.

By using the natural boundary condition Eq. (3.16) with displacement (w) relation in Eq. (3.17-3.20), the bending rotation  $\phi_y$ , shear force  $V_x$ , and moment  $M_{xx}$  is obtained and are given by Eq. (3.23-3.25).

$$\phi_{y_m}(x,y) = \Phi_{y_m}(x)\sin(\alpha_m y) = \frac{dW_m(x)}{dx}\sin(\alpha_m y)$$

$$= -((A_m r_{1_m} \sinh(r_{1_m} x) + B_m r_{1_m} \cosh(r_{1_m} x) + C_m r_{2_m} \sin(r_{2_m} x) + D_m r_{2_m} \cos(r_{2_m} x))\sin(\alpha_m y)$$
(3.23)

$$V_{x_m}(x,y) = V_{x_m}(x)\sin(\alpha_m y)$$

$$= -D_{\text{FGM}}\left(A_m\left(r_{1m}^3 - (2-\nu)\alpha_m^2 r_{1m}\right)\sinh(r_{1m}x) + B_m\left(r_{1m}^3 - (2-\nu)\alpha_m^2 r_{1m}\right)\cosh(r_{1m}x) + C_m\left(r_{2m}^3 + (2-\nu)\alpha_m^2 r_{2m}\right)\sin(r_{2m}x) + D_m\left(r_{2m}^3 + (2-\nu)\alpha_m^2 r_{2m}\right)\cos(r_{2m}x)\right)\sin(\alpha_m y)$$
(3.24)

$$M_{xx_{m}}(x,y) = \mathcal{M}_{xx_{m}}(x)\sin(\alpha_{m}y)$$

$$= -D_{FGM}\left(A_{m}\left(r_{1_{m}}^{2} - \nu\alpha_{m}^{2}\right)\cosh\left(r_{1_{m}}x\right) + B_{m}\left(r_{1_{m}}^{2} - \nu\alpha_{m}^{2}\right)\sinh\left(r_{1_{m}}x\right) + C_{m}\left(r_{2,m}^{2} - \nu\alpha_{m}^{2}\right)\cos(r_{2m}x) + D_{m}\left(r_{2_{m}}^{2} - \nu\alpha_{m}^{2}\right)\sin(r_{2m}x)\right)\sin(\alpha_{m}y)$$
(3.25)

The boundary conditions of displacements are given by

$$x = 0$$
  $W_m = W_1;$   $\phi_{y_m} = \phi_{y_1}$   $x = b$   $W_m = W_2;$   $\phi_{y_m} = \phi_{y_2}$  (3.26)

The force boundary conditions are given by

$$x = 0$$
  $V_{xm} = -V_1;$   $M_{xx_m} = -M_1$   $x = b$   $V_{xm} = V_2;$   $M_{xx_m} = M_2$  (3.27)

The displacement boundary conditions present in Eq. (3.26) are substituted into solution Eq. (3.20) and bending rotation Eq. (3.23) to develop the following matrix.

$$\begin{bmatrix} W_1 \\ \Phi_{y_1} \\ W_2 \\ \Phi_{y_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -r_{1_{1m}} & 0 & -r_{2m} \\ C_{h_1} & S_{h_1} & C_2 & S_2 \\ -r_{1_m} S_{h_1} & -r_{1_m} C_{h_1} & r_{2_m} S_2 & -r_{2_m} C_2 \end{bmatrix} \begin{bmatrix} A_m \\ B_m \\ C_m \\ D_m \end{bmatrix}$$
 (3.28)

or,

$$\boldsymbol{\delta} = \boldsymbol{AC},\tag{3.29}$$

where

$$C_{h_i} = \cosh(r_i b), \quad S_{h_i} = \sinh(r_i b)$$

$$C_i = \cos(r_i b), \quad S_i = \sin(r_i b) \text{ with } i = 1, 2.$$

$$(3.30)$$

For force boundary conditions, a similar procedure is to be followed, i.e., Eq. (3.27) is substituted into shear force Eq. (3.24) and moment Eq. (3.25) to obtain the following matrix.

$$\begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} 0 & R_1 & 0 & R_2 \\ L_1 & 0 & L_2 & 0 \\ -R_1 S_{h_1} & -R_1 C_{h_1} & R_2 S_2 & -R_2 C_2 \\ -L_1 C_{h_1} & -L_1 S_{h_1} & -L_2 C_2 & -L_2 S_2 \end{bmatrix} \begin{bmatrix} A_m \\ B_m \\ C_m \\ D_m \end{bmatrix}$$
(3.31)

or,

$$\mathbf{F} = \mathbf{RC} \tag{3.32}$$

where

$$R_i = D_{FGM}[r_{im}^3 - \alpha^2 r_{im}(2 - \nu)], L_i = D_{FGM}(r_{im}^3 - \alpha^2 \nu)$$
(3.33)

with i = 1,2.

For excluding the constant vector value of C, the following relationship can be formed as

$$\mathbf{F} = \mathbf{K}\boldsymbol{\delta} \tag{3.34}$$

where

$$K = RA^{-1} \tag{3.35}$$

In Eq. (3.35), K indicates the square 4×4 symmetric dynamic stiffness (DS) matrix, including the independent terms ( $S_{vv}$ ,  $S_{vm}$ ,  $F_{vv}$ ,  $F_{vm}$ ,  $S_{mm}$ ,  $S_{vn}$ ). Therefore, the generate DS matrix of the single plate element can be expressed as

$$[K] = \begin{bmatrix} S_{vv} & S_{vm} & F_{vv} & F_{vm} \\ S_{vm} & S_{mm} & -F_{vm} & F_{mm} \\ F_{vv} & -F_{vm} & S_{vv} & -S_{vm} \\ F_{vm} & F_{mm} & -S_{vm} & S_{mm} \end{bmatrix}$$
(3.36)

The mathematical expressions of Eq. (3.36) are explained in Appendix B.

### 3.2.7. Formulation of DS matrix assembly method with boundary conditions

The formulation of the DS matrix by DSM for a single element system in Eq. (3.36) is represented in Fig.3.5. The development of the DS matrix for the overall system is followed the same methodological assembly process of Eq. (3.36), and this process is very similar to FEM. The only difference from FEM is that the line is taken rather than the point node for each strip connection of plate element, as present in Fig.3.5.

Fig. 3.5: Schematic diagram of global assembly procedure of dynamic stiffness matrix

Now, the penalty method is implemented with the levy type solution in the global matrix where a substantial degree of freedom (DOF) of the P-FGM plate edges is suppressed. It is obtained by applying a very large value of stiffness to a relevant term of the leading diagonal of the global DS matrix. The boundary conditions of the P-FGM plate are employed at the contrary sides, which is the generalized form of simple supported, free, and clamped.

For applying the boundary conditions, the penalty method follows a defined pattern and can be summarized as below:

- Displacement  $(W_i)$  is penalized for simply supported edge conditions.
- Displacement  $(W_i)$  and rotation  $(\phi_i)$  are penalized for clamped edge conditions
- No penalty is applied for the free edge condition.

where i represent the suppressed node.

### 3.2.8. Extraction of natural frequency using Wittrick and Williams algorithm

For the whole structure, an overall DS matrix is developed with relevant, suitable constraints used to carry out the natural frequency of the P-FGM plate. The algorithm of Wittrick and Williams [14] is applied to extract the eigenvalues of the plate, and it solves the transcendental behavior of the overall DS matrix, which ensures that no natural frequencies are missed out [70-74], in the plate. The process methodology of W-W algorithm is represented by the flow chart and is as shown in Fig.3.6.

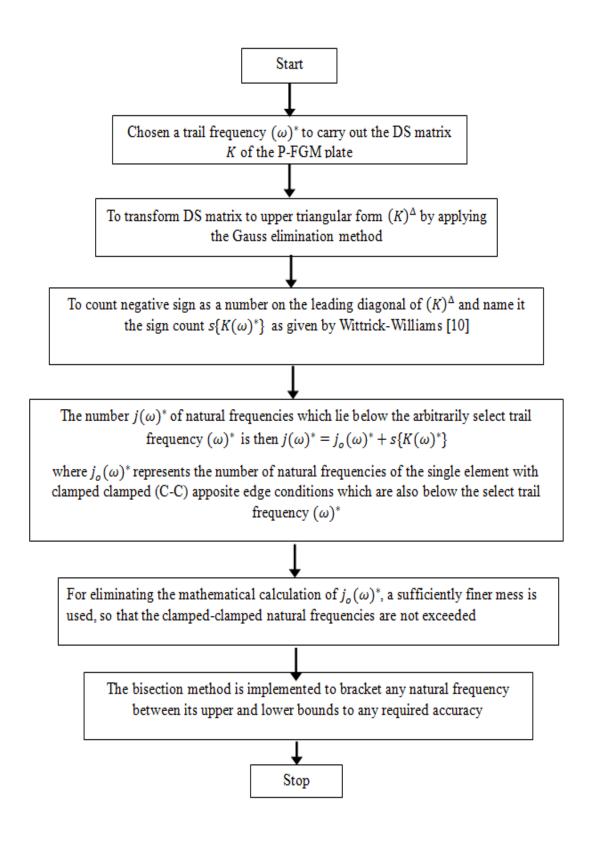


Fig. 3.6: Flow chart representation of the W-W algorithm

#### 3.3. Results and discussions

The formulation of DSM explained above has been employed in the software program MATLAB to carry out the free vibrtaion frequencies and mode shape of the P-FGM plate. The notation C, F, and S represent the edge conditions of the FGM plate, such as clamped, free, and simply supported, respectively. Its means that the edge conditions SCSF stands as simply supported (at y = 0 and y = a), clamped (at x = 0) and free (at x = b) for a given P-FGM plate. The symbols, m, and n denote a particular frequency mode shape, where m represents the number of half-sine waves in a particular x direction and m represents the nth lower frequency for a corresponding value of m.

A qualitative comparative study is reported in this section, first, the eigenvalues obtained by DSM are compared with those available in published literature shows very good agreement with them. Later, the design parameters effects (material gradient index, elastic modulus, material properties ratio, and aspect ratio) on natural frequencies are highlighted in the form of tables and graphs.

#### 3.3.1 Comparative study

#### 3.3.1.1 Comparison of results of P-FGM plate with or without elastic foundation

Here, the following mathematical nondimensional natural frequency parameters have been used for effective comparison of natural frequencies obtained by DSM and can be expressed as [101,114].

$$(\omega^{\hat{}} = \omega \frac{a^2}{h} \sqrt{\rho_m/E_m}), \widetilde{\omega} = \omega h \sqrt{\rho_c/E_c}, \ \omega^* = \omega a^2 \sqrt{\frac{\rho_c h}{D_c}}, \ K_p = \frac{k_p a^4}{D_c}, \ K_w = \frac{k_w a^2}{D_c}$$
 (3.37)

At a very low value of material gradient index (p = 0), the inhomogeneous power-law FGM plate changes to homogeneous in nature and is called an isotropic plate. The DSM frequency results for an isotropic rectangular plate with SSSS boundary condition are compared with published results and shown in Table.3.2. The natural frequency results computed by the DSM method are in excellent agreement and exactly match up to four decimal values of Ref. [181] as seen in Table 3.2.

**Table 3.2:** Comparison of frequencies parameters  $(\omega^* = \omega a^2 \sqrt{\rho_c h/D_c})$  of SSSS homogeneous square isotropic plate.

Course	o/b	Mode	Mode	Mode	Mode	Mode	Mode
Source	a/b	(1 1)	(1 2)	(2 1)	(2 2)	(13)	(3 1)
Ref. [122]		11.4487	16.1862	24.0818	35.1357	41.0575	45.7950
Ref. [181]	0.4	11.4487	16.1862	24.0818	35.1358	41.0576	45.7949
Present		11.4487	16.1862	24.0818	35.1358	41.0576	45.7950
Ref. [122]		14.256	27.4156	43.8649	49.3480	57.0243	78.9568
Ref. [181]	2/3	14.2561	27.4156	43.8649	49.3480	57.0244	78.9568
Present		14.2561	27.4156	43.8649	49.3480	57.0244	78.9568
Ref. [122]		19.7392	49.3480	49.3480	78.9568	98.6960	98.6960
Ref. [11]	1	19.7392	49.3480	49.3480	78.9568	98.6960	98.6960
Ref. [181]	1	19.7392	49.3480	49.3480	78.9568	98.6960	98.6960
Present		19.7392	49.3480	49.3480	78.9568	98.6960	98.6960
Ref. [122]		32.0762	61.6850	98.6960	111.033	128.3048	177.6528
Ref. [181]	1.5	32.0762	61.6850	98.6960	111.033	128.3049	177.6529
Present		32.0762	61.6850	98.6960	111.033	128.3049	177.6529
Ref. [122]		71.5564	101.1634	150.5114	219.5986	256.6097	286.2185
Ref. [181]	2.5	71.5546	101.1634	150.5115	219.5987	256.6097	286.2185
Present		71.5564	101.1634	150.5115	219.5987	256.6097	286.2185

**Table 3.3:** Comparison of the DSM natural frequencies  $(\omega^{\hat{}} = \omega \frac{a^2}{h} \sqrt{\rho_m/E_m})$  with available literature values for FGM plates.

			SS	SS		SCSC			
Mode	Source	p = 0	p = 0.5	p = 1.0	p = 2.0	p = 0	p = 0.5	p = 1.0	p = 2.0
1	Ref. [122]	11.7460	9.9456	8.9618	8.1478	17.2260	14.5870	13.1440	11.9500
	Ref. [124]	11.7460	9.9456	8.9618	8.1478	17.2270	14.5870	13.1440	11.9500
	Ref. [139]	11.7460	9.9464	8.9618	8.1486	17.2260	14.5870	13.1420	11.9500
	Present	11.7466	9.9465	8.9627	8.1486	17.2284	14.5881	13.1452	11.9513
2	Ref. [122]	29.3610	24.8620	22.4030	20.3680	32.5680	27.5770	24.8490	22.5920
	Ref. [124]	29.3610	24.8610	22.4020	20.3670	32.5710	27.5780	24.8510	22.5930
	Ref. [139]	29.3670	24.8660	22.4070	20.3720	32.5750	27.5840	24.8550	22.5980
	Present	29.3666	24.8661	22.4066	20.3716	32.5771	27.5847	24.8563	22.5987
3	Ref. [122]	29.3610	24.8620	22.4030	20.3680	41.2480	34.9270	31.4720	28.6140

	Ref. [124]	29.3610	24.8610	22.4020	20.3670	41.2480	34.9260	31.4710	28.6120
	Ref. [139]	29.3670	24.8660	22.4070	20.3720	41.2560	34.9330	31.4780	28.6190
	Present	29.3666	24.8661	22.4066	20.3716	41.2559	34.9334	31.4782	28.6192
4	Ref. [122]	46.9660	39.7680	35.8350	32.5800	56.2560	47.6350	42.9230	39.0250
	Ref. [124]	46.9710	39.7730	35.8390	32.5840	56.2690	47.6450	42.9320	39.0310
	Ref. [139]	46.9860	39.7860	35.8500	32.5940	56.2870	47.6610	42.9470	39.0460
	Present	46.9865	39.7858	35.8506	32.5945	56.2869	47.6609	42.9468	39.0462
	Ref. [122]	58.7220	49.7230	44.8040	40.7350	60.8100	51.4910	46.3980	42.1840
5	Ref. [124]	58.7120	49.7140	44.7980	40.7290	60.8070	51.4880	46.3940	42.1780
	Ref. [139]	58.7330	49.7320	44.8130	40.7430	60.7980	51.4800	46.3880	42.1750
	Present	58.7332	49.7323	44.8133	40.7431	60.8280	51.5061	46.4116	42.1963

**Table 3.4:** Comparison of frequencies parameters ( $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$ ) of square P-FGM simply supported plate for different elastic modulus values and different values of material gradient index for h/a = 0.05.

$K_{w}$	$K_P$	Source		Mate	rial gradien	t index (p)	
$\kappa_{W}$	$\kappa_P$	Source	0	0.5	1	2	5
0	0	Ref. [209]	0.0291	0.0249	0.0227	0.0209	0.0197
		Ref. [238]	0.0291	0.0246	0.0222	0.0222	0.0191
		Ref. [267] (2D)	0.0291	0.0246	0.0222	0.0222	0.0191
		Ref. [267] (quasi-3D)	0.0291	0.0248	0.0226	0.0207	0.0195
		Present	0.0291	0.0246	0.0227	0.0209	0.0197
0	100	Ref. [209]	0.0406	0.0389	0.0382	0.0380	0.0381
		Ref. [238]	0.0406	0.0386	0.0378	0.0374	0.0377
		Ref. [267] (2D)	0.0406	0.0386	0.0378	0.0374	0.0377
		Ref. [267] (quasi-3D)	0.0406	0.0387	0.0380	0.0376	0.0378
		Present	0.0406	0.0389	0.0382	0.0380	0.0381
100	0	Ref. [209]	0.0298	0.0258	0.0238	0.0221	0.0210
		Ref. [238]	0.0298	0.0255	0.0233	0.0214	0.0204
		Ref. [267] (2D)	0.0298	0.0255	0.0233	0.0214	0.0204
		Ref. [267] (quasi-3D)	0.0298	0.0257	0.0233	0.0219	0.0208
		Present	0.0298	0.0255	0.0233	0.0214	0.0204

100	100	Ref. [209]	0.0411	0.0395	0.0388	0.0386	0.0388
		Ref. [238]	0.0411	0.0392	0.0384	0.0381	0.0384
		Ref. [267] (2D)	0.0411	0.0392	0.0384	0.0381	0.0384
		Ref. [267] (quasi-3D)	0.0411	0.0393	0.0386	0.0383	0.0385
		Present	0.0411	0.0392	0.0384	0.0381	0.0384

The fundamental frequencies of square P-FGM plate for different material gradient index, p with and without elastic foundation obtained by DSM are compared with those published in the literature in Tables 3.3 and 3.4. The results obtained by DSM method are in excellent agreement.

### 3.3.2 Extraction of the natural frequency of P-FGM plate supported on Winkler-Pasternak foundation

This section highlights the extraction of a new set of natural frequencies for the P-FGM plate by DSM under two parametric geometric configurations (one square and another rectangular) supported on Winkler and Pasternak elastic foundation.

### 3.3.2.1. Extraction of the natural frequency of a square P-FGM plate with Winkler-Pasternak foundation

In this subsection, a new set of natural frequency results of a square isotropic plate (p=0) for a different combination of Winkler modulus  $(K_w=0,100,1000)$  with Levy type edge conditions are present in Table 3.5. It is noticed that at a particular mode, the natural frequency increases with increases the Winkler modulus value  $(K_w=0,100,1000)$ . This is because of stiffness of the isotropic plate increases. This table shows that the maximum natural frequency is obtained for SCSC boundary conditions and the minimum for SFSF boundary conditions. The reason is that by adding more constraint at the SCSC edge condition, the stiffness of the plate increases and gives the maximum value of natural frequency while for fee edge condition SFSF, removing the constraints at the particular edge which reduces the plate stiffness and gives the minimum value of natural frequency. Similarly, Table 3.6 shows the natural frequency results for Pasternak modulus  $(K_P=0,100,1000)$  for the square isotropic plate under Levy type edge conditions, and natural frequency increases as the Pasternak modulus  $(K_P=0,100,1000)$  of the plate, because of increase in the plate stiffness. It is noticed from Tables (3.5-3.7) that the effect of the

Pasternak modulus (at  $K_w = 0$ ) is higher than the Winkler modulus (at  $K_P = 0$ ).

The fundamental natural frequencies of power-law FGM plate supported on different combinations of Winkler and Pasternak modulus with different values of material gradient index (p) under six boundary conditions are reported in Table 3.7. It can be observed from the table that the frequency of the plate decreases as increases the material gradient index (p) for a given edge condition and a particular combination of foundation value. As the material gradient increases (p), the P-GM plate stiffness decreases, which means adding more metal constituents to the plate. The mode shapes for square power law FGM plate for elastic foundation  $(K_w, K_P = 100,100)$  for material gradient index (p = 1) with SSSS boundary condition are present in Fig.3.7. The developed mode shapes for the P-FGM plate are the same as observed in an isotropic plate.

**Table 3.5:** Fundamental frequencies parameters ( $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$ ) for square isotropic P-FGM plate under Levy type boundary conditions for different value of Winkler parameters for  $K_p = 0$  and h/a = 0.01.

-				Mo	de no.		
BCs	$K_w$	1	2	3	4	5	6
SCSC	0	28.9509	54.7431	69.3270	94.5853	102.2162	129.0955
	$10^{2}$	30.6293	55.6489	70.0445	95.1124	102.7042	129.4823
	$10^{3}$	42.8737	63.2203	76.1987	99.7315	106.9960	132.9122
SCSS	0	23.6463	51.6743	58.6464	86.1345	100.2698	113.2281
	$10^{2}$	25.6739	52.6330	59.4928	86.7130	100.7672	113.6688
	$10^{3}$	39.4861	60.5824	66.6288	91.7559	105.1382	117.5611
SSSS	0	19.7392	49.3480	49.3480	78.9568	98.6960	98.6960
	$10^{2}$	22.1277	50.3510	50.3510	79.5876	99.2014	99.2014
	$10^{3}$	37.2778	58.6108	58.6108	85.0540	103.6384	103.6384
SCSF	0	12.6874	33.0651	41.7019	63.0148	72.3976	90.6114
	$10^{2}$	16.1545	34.5442	42.8842	63.8034	73.0849	91.1615
	$10^{3}$	34.0730	45.7526	52.3359	70.5044	79.0026	95.9709
SSSF	0	11.6845	27.7563	41.1967	59.0655	61.8606	90.2941
	$10^{2}$	15.3795	29.5028	42.3930	59.9060	62.6637	90.8461
	$10^{3}$	33.7124	42.0763	51.9342	66.9980	69.4747	95.6714

SFSF	0	9.6314	16.1348	36.7256	38.9450	46.7381	70.7401
	$10^{2}$	13.8839	18.9824	38.0627	40.2083	47.7960	71.4434
	$10^{3}$	33.0570	35.5011	48.4641	50.1668	56.4310	77.4865

**Table 3.6:** Fundamental frequencies parameters ( $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$ ) for square isotropic P-FGM plate under Levy type boundary conditions for different value of Pasternak parameters for  $K_W = 0$  and h/a = 0.01.

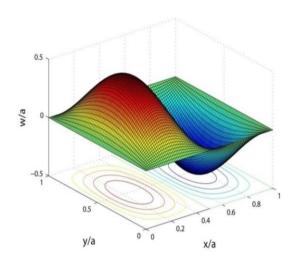
				Mod	e no.		
BCs	$K_p$	1	2	3	4	5	6
SCSC	0	28.9509	54.7431	69.3270	94.5853	102.2162	129.0955
	$10^{2}$	54.6812	89.8831	101.4516	131.6468	142.9568	165.7549
	$10^{3}$	146.7424	230.7013	239.9439	301.8570	331.5625	349.4051
SCSS	0	23.6463	51.6743	58.6464	86.1345	100.2698	113.2281
	$10^{2}$	51.3212	87.6587	93.0892	124.7830	141.3698	152.2442
	$10^{3}$	144.2127	229.0622	233.5423	296.6795	330.3814	339.0439
SSSS	0	19.7392	49.3480	49.3480	78.9568	98.6960	98.6960
	$10^{2}$	48.6164	85.8489	85.8489	118.8691	140.0375	140.0375
	$10^{3}$	48.6164	85.8489	85.8489	118.8691	140.0375	140.0375
SCSF	0	12.6874	33.0651	41.7019	63.0148	72.3976	90.6114
	$10^{2}$	51.5156	59.4597	88.5274	94.6425	128.3916	143.6038
	$10^{3}$	144.2145	229.0713	233.5621	296.7335	330.4089	339.1497
SSSF	0	11.6845	27.7563	41.1967	59.0655	61.8606	90.2941
	$10^{2}$	48.7568	59.4596	86.5442	87.0169	121.8683	141.9200
	$10^{3}$	141.8777	227.5675	227.5764	291.9232	329.3228	329.3885
SFSF	0	9.6314	16.1348	36.7256	38.9450	46.7381	70.7401
	$10^{2}$	48.9006	59.5157	87.3103	88.2736	125.2363	144.2855
	$10^{3}$	141.8793	227.5757	227.5935	291.9714	329.3480	329.4796

**Table 3.7:** Fundamental frequency parameter  $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$  for square P-FGM plate for different Winkler-Pasternak parameters with different values of p with different Levy type boundary conditions and h/a = 0.01.

BC's	$K_w$	$K_P$	p = 0	p = 0.1	p = 0.5	<i>p</i> = 1	<i>p</i> = 2	<i>p</i> = 5	p = 10
SCSC	0	0	28.9509	27.8514	24.5141	22.0894	20.0832	19.0419	18.4395
	$10^{2}$	0	30.6293	29.4661	25.9353	23.3700	21.2475	20.1459	19.5085
	0	$10^{2}$	54.6812	52.6046	46.3013	41.7216	37.9323	35.9656	34.8278
	$10^2$	$10^2$	55.5881	53.4771	47.0692	42.4136	38.5614	36.5621	35.4054
SCSS	0	0	23.6463	22.7483	20.0225	18.0421	16.4034	15.5530	15.0609
	$10^{2}$	0	25.6739	24.6989	21.7393	19.5891	17.8099	16.8865	16.3523
	0	$10^{2}$	51.3212	49.3722	43.4562	39.1579	35.6015	33.7556	32.6878
	$10^2$	$10^2$	52.2864	50.3008	44.2735	39.8944	36.2710	34.3905	33.3025
SSSS	0	0	19.7392	18.9896	16.7142	15.0610	13.6931	12.9831	12.5724
	$10^{2}$	0	22.1277	18.7366	16.8834	15.3500	14.5541	14.5541	14.0937
	0	$10^{2}$	48.6164	46.7702	41.1659	37.0942	33.7252	31.9766	30.9650
	$10^2$	$10^{2}$	49.6342	47.7493	42.0277	37.8708	34.4312	32.6460	31.6133
SCSF	0	0	12.6874	12.2055	10.7430	9.6804	8.8012	8.3449	8.0809
	$10^2$	0	16.1545	15.5411	13.6788	12.3259	11.2064	10.6254	10.2892
	0	$10^{2}$	51.5156	49.5593	43.6208	39.3063	35.7363	33.8835	32.8116
	$10^{2}$	$10^{2}$	52.4772	50.4844	44.4351	40.0400	36.4034	34.5160	33.4241
SSSF	0	0	11.6845	11.2408	9.8939	8.9153	8.1055	7.6853	7.4422
	$10^2$	0	15.3795	14.7954	13.0226	11.7345	10.6687	10.1156	9.7956
	0	$10^{2}$	48.7568	46.9052	41.2848	37.2013	33.8225	32.0689	31.0544
	$10^{2}$	$10^{2}$	49.7717	47.8816	42.1442	37.9757	34.5266	32.7365	31.7009
SFSF	0	0	9.6314	9.2656	8.1554	7.3487	6.6813	6.3349	6.1345
	$10^{2}$	0	13.8839	13.3567	11.7562	10.5934	9.6313	9.1319	8.8430
	0	$10^{2}$	48.9006	47.0436	41.4066	37.3110	33.9223	32.1635	31.1460
	$10^{2}$	$10^{2}$	49.9126	48.0171	42.2635	38.0832	34.6243	32.8292	31.7906

Mode shapes of randomly selected modes of the square FGM plates for all six Levy type boundary conditions are shown in Fig. 3.7. The mode shapes of  $2^{nd}$  mode and  $7^{th}$  mode of SSSS and SFSS square P-FGM plates are shown in Fig. 3.7 (a) and (b), whereas the mode shapes of  $5^{th}$  mode of SCSS, SFSF and SCSF square plate are plotted in Fig. 3.7 (c), (d) and (e), respectively. The mode shape of  $6^{th}$  mode of SCSC square plate is shown in Fig. 3.7 (f). Note that the variation of p in the power-law influences the natural frequencies of the plates however does not influence the mode shapes. Thus the mode shapes of the P-FGM plates

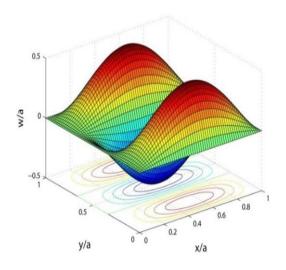
remain unchanged due to the variation of p including the special case of the homogeneous isotropic plate (p = 0).

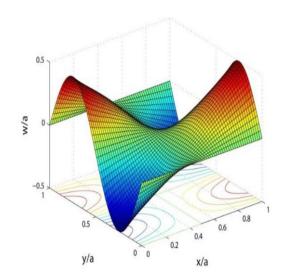


0.5 0.5 0.5 0.6 0.8 0.8 0.8 0.8 0.8

(a) SSSS (2nd mode i.e. *m*=1, *n*=2)

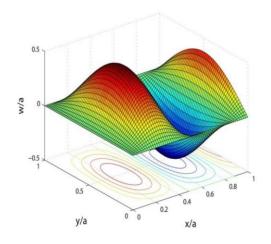
(b) SFSS (7th mode i.e. m=2, n=3)

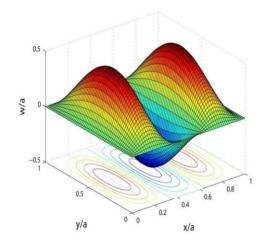




(c) SCSS (5th mode i.e. m=3, n=1)

(d) SFSF (9th mode i.e. m=2, n=2)





**Fig. 3.7:** Mode shapes for square plates with all six Levy type boundary conditions: **(a) SSSS** (2<sup>nd</sup> mode i.e. m=1, n=2), **(b) SFSS** (7<sup>th</sup> mode i.e. m=2, n=3), **(c) SCSS** (5<sup>th</sup> mode i.e. m=3, n=1), **(d) SFSF** (5<sup>th</sup> mode i.e. m=2, n=2), **(e) SCSF** (5<sup>th</sup> mode i.e. m=1, n=3) **(f) SCSC** (6<sup>th</sup> mode i.e. m=1, n=3)

## 3.3.2.2. Extraction of the natural frequency of rectangular P-FGM plate supported on Winkler-Pasternak elastic foundation

In this subsection, the fundamental natural frequencies of the P-FGM rectangular plate obtained by DSM are reported. Here, Table (3.8-3.13) represent the fundamental frequency results for three different value of aspect ratios for different value of material gradient index (p) for Levy type edge conditions with different combination of elastic foundation  $(K_w, K_P)$ .

It is illustrating from these tables that the plates frequency decreases with an increase in the material gradient index (p) for a particular value of elastic foundation and aspect ratio. Besides this, as the aspect ratio increases with elastic foundation, the plate's frequency increases for a given value of material gradient index under all boundary conditions except the SFSF condition. As the aspect ratio increases with a change in boundary condition under an elastic foundation, natural frequency increases because the plate's stiffness increases. Expect in SFSF boundary condition (no constraint) as aspect ratio increases, the stiffness of the plate decreases because of this natural frequency decreases as seen in Table 3.13.

**Table 3.8:** Fundamental frequency parameter  $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$  for SCSC rectangular P-FGM plate with different combination of  $(K_w, K_P, p)$  and h/a = 0.01.

a/b	$(K_w,K_P)$	p = 0	p = 0.1	p = 0.5	p = 1	p = 2	p = 5	<i>p</i> = 10
0.5	(0,0)	13.6857	13.1660	11.5884	10.4422	9.4938	9.0015	8.7168
	$(10^2, 10^2)$	39.3656	37.8706	33.3328	30.0358	27.3078	25.8920	25.0729
	$(10^3, 10^3)$	116.8544	112.4168	98.9464	89.1596	81.0618	76.8589	74.4275
1.5	(0,0)	57.3597	55.1815	48.5693	43.7653	39.7904	37.7273	36.5338
	$(10^2, 10^2)$	84.3726	81.1684	71.4424	64.3760	58.5291	55.4946	53.7390
	$(10^3, 10^3)$	199.5002	191.9240	168.9266	152.2181	138.3930	131.2177	127.0666
2	(0,0)	95.2625	91.6448	80.6634	72.6850	66.0834	62.6572	60.6750
	$(10^2, 10^2)$	122.4423	117.7924	103.6779	93.4232	84.9381	80.5343	77.9866
	$(10^3, 10^3)$	256.1198	246.3934	216.8692	195.4188	177.6700	168.4583	163.1291

**Table 3.9:** Fundamental frequency parameter  $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$  for SCSS rectangular P-FGM plate with different combinations of  $(K_w, K_P, p)$  and h/a = 0.01.

a/b	$(K_w, K_P)$	p = 0	p = 0.1	p = 0.5	p = 1	p = 2	<i>p</i> = 5	<i>p</i> = 10
0.5	(0,0)	12.9185	12.4279	10.9387	9.8568	8.9615	8.4969	8.2281
	$(10^2, 10^2)$	38.9298	37.4514	32.9637	29.7033	27.0055	25.6054	24.7953
	$(10^3, 10^3)$	211.6327	203.5958	179.1998	161.4752	146.8093	139.1977	134.7942
1.5	(0,0)	42.5965	40.9788	36.0685	32.5010	29.5491	28.0171	27.1307
	$(10^2, 10^2)$	73.5335	70.7410	62.2644	56.1058	51.0101	48.3653	46.8353
	$(10^3, 10^3)$	191.0547	183.7992	161.7754	145.7742	132.5344	125.6628	121.6875
2	(0,0)	69.3270	66.6942	58.7025	52.8963	48.0920	45.5986	44.1561
	$(10^2, 10^2)$	101.9432	98.0718	86.3203	77.7824	70.7178	67.0513	64.9302
	$(10^3, 10^3)$	242.0187	232.8278	204.9291	184.6597	167.8881	159.1836	154.1478

**Table 3.10:** Fundamental frequency parameter  $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$  for SSSS rectangular P-FGM plate with different combinations of  $(K_w, K_P, p)$  and h/a = 0.01.

a/b	$(K_w, K_P)$	p = 0	p = 0.1	p = 0.5	p = 1	p = 2	<i>p</i> = 5	<i>p</i> = 10
0.5	(0,0)	12.3370	11.8684	10.4463	9.4131	8.5581	8.1144	7.8577

	$(10^2, 10^2)$	38.5474	37.0835	32.6399	29.4115	26.7402	25.3538	24.5518
	$(10^3, 10^3)$	116.1430	111.7324	98.3440	88.6168	80.5682	76.3910	73.9744
1.5	(0,0)	32.1206	30.9008	27.1981	24.5080	22.2820	21.1268	20.4584
	$(10^2, 10^2)$	65.9075	63.4046	55.8071	50.2873	45.7200	43.3495	41.9781
	$(10^3, 10^3)$	184.8037	177.7856	156.4824	141.0048	128.1981	121.5514	117.7061
2	(0,0)	49.3480	47.4739	41.7853	37.6524	34.2326	32.4577	31.4309
	$(10^2, 10^2)$	86.4293	83.1470	73.1839	65.9453	59.9559	56.8473	55.0490
	$(10^3, 10^3)$	229.7460	221.0212	194.5372	175.2956	159.3745	151.1114	146.3310

**Table 3.11:** Fundamental frequency parameter  $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$  for SFSC rectangular P-FGM plate with different combinations of  $(K_w, K_P, p) h/a = 0.01$ .

a/b	$(K_w, K_P)$	p = 0	p = 0.1	p = 0.5	p = 1	p = 2	p = 5	p = 10
0.5	(0,0)	10.4254	10.0295	8.8277	7.9546	7.2321	6.8571	6.6402
	$(10^2, 10^2)$	38.9438	37.4648	32.9756	29.7140	27.0152	25.6146	24.8043
	$(10^3, 10^3)$	116.4909	112.0671	98.6386	88.8823	80.8096	76.6198	74.1960
1.5	(0,0)	16.8377	16.1982	14.2573	12.8471	11.6802	11.0747	10.7243
	$(10^2, 10^2)$	74.6725	71.8367	63.2288	56.9749	51.8002	49.1145	47.5607
	$(10^3, 10^3)$	191.0654	183.8095	161.7844	145.7824	132.5418	125.6699	121.6943
2	(0,0)	22.8154	21.9490	19.3189	17.4081	15.8270	15.0064	14.5317
	$(10^2, 10^2)$	106.1927	102.1600	89.9186	81.0248	73.6658	69.8464	67.6368
	$(10^3, 10^3)$	242.0646	232.8720	204.9680	184.6947	167.9199	159.2138	154.1770

**Table 3.12:** Fundamental frequency parameter  $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$  for SSSF rectangular P-FGM plate with different combinations of  $(K_w, K_P, p)$  and h/a = 0.01.

a/b	$(K_w,K_P)$	p = 0	p = 0.1	p = 0.5	p = 1	p = 2	p = 5	p = 10
0.5	(0,0)	10.2991	9.9080	8.7208	7.8582	7.1445	6.7740	6.5597
	$(10^2, 10^2)$	38.5594	37.0951	32.6501	29.4207	26.7486	25.3618	24.5594
	$(10^3, 10^3)$	116.1431	111.7325	98.3441	88.6169	80.5683	76.3911	73.9744
1.5	(0,0)	13.7178	13.1969	11.6155	10.4666	9.5160	9.0226	8.7372

	$(10^2, 10^2)$	66.6100	64.0804	56.4019	50.8232	46.2073	43.8115	42.4256
	$(10^3, 10^3)$	184.8124	177.7939	156.4897	141.0113	128.2041	121.5571	117.7116
2	(0,0)	16.1347	15.5220	13.6621	12.3107	11.1926	10.6123	10.2766
	$(10^2, 10^2)$	60.3499	58.0581	51.1012	46.0468	41.8646	39.6941	38.4383
	$(10^3, 10^3)$	229.7798	221.0537	194.5659	175.3214	159.3980	151.1336	146.3525

**Table 3.13:** Fundamental frequency parameter  $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$  for SFSF rectangular P-FGM plate with different combinations of  $(K_w, K_P, p) h/a = 0.01$ .

a/b	$(K_w, K_P)$	p = 0	p = 0.1	p = 0.5	p = 1	p = 2	p = 5	<i>p</i> = 10
0.5	(0,0)	9.7362	9.3664	8.2441	7.4287	6.7540	6.4038	6.2012
	$(10^2, 10^2)$	38.5715	37.1067	32.6604	29.4300	26.7570	25.3697	24.5672
	$(10^3, 10^3)$	116.14329	111.7326	98.3442	88.6170	80.5684	76.3912	73.9745
1.5	(0,0)	9.5580	9.1950	8.0932	7.2927	6.6303	6.2866	6.0877
	$(10^2, 10^2)$	59.5674	57.3052	50.4386	45.4497	41.3218	39.1794	37.9399
	$(10^3, 10^3)$	184.8210	177.8022	156.4970	141.0179	128.2101	121.5628	117.7171
2	(0,0)	9.5124	9.1512	8.0546	7.2579	6.5987	6.2566	6.0587
	$(10^2, 10^2)$	57.7866	55.5921	48.9308	44.0910	40.0865	38.0081	36.8057
	$(10^3, 10^3)$	229.8137	221.0863	194.5945	175.3472	159.4214	151.1559	146.3741

#### 3.3.3 A parametric study

In this section, the effect of the different values of geometrical parameters such as density ratio, modulus, aspect ratio, and Winkler and Pasternak modulus on the natural frequency of the P-FGM plate is highlighted.

The variations of Winkler and Pasternak elastic modulus  $(K_w, K_P)$  for different modulus ratio  $(E_{rat})$  with a fixed value of density ratio  $(\rho_{rat} = 2)$  and material gradient index (p = 2) on the natural frequency of square P-FGM plate under SCSC edge condition are shown in Fig. 3.8 (a-d). It can be noticed from these figures that at a given mode, the plate's natural frequency is decreases as Young's modulus ratio  $(E_{rat})$  increases because plate stiffness decreases due to added more metal constituents in the P-FGM plate. The variation trend of

the P-FGM plate's natural frequency is the same as obtained by different authors [259-260]. It is also noticed that the effect of the Pasternak modulus on natural frequency for a particular mode is significantly large than the Winkler modulus, and similar attention is also reported by authors [259-260]. For the low value of Young's modulus ratio ( $E_{rat} < 10$ ), the decrease of the natural frequency of a particular mode with a fixed value of elastic modulus is more evident.

The variations of density ratio ( $\rho_{rat}$ ) for a different combination of elastic modulus ( $K_w, K_P$ ) with a fixed value of Young's modulus ratio ( $E_{rat} = 2$ ) and material gradient index (p = 2) on the natural frequency of square P-FGM plate under SCSC edge condition are reported in Fig. 3.9 (a-c). As the density ratio increases ( $\rho_{rat}$ ), the frequency of the P-FGM plate is increased, and its shows that for the lower value of density ratio ( $\rho_{rat} < 10$ ), increase in natural frequencies are significant for a particular value of Winkler and Pasternak elastic modulus. Again, it shows that the Pasternak modulus's effect on natural frequency is significant than the Winkler modulus, and similar observations are reported by authors [46,47,56].

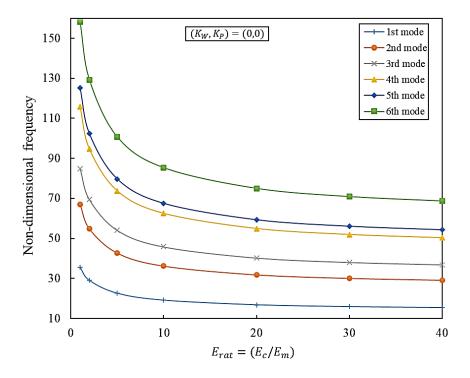
The effects of elastic modulus  $(K_{w}, K_{P})$  under the different values of material gradient index (p) and different values of  $E_{rat} = \rho_{rat}$  on the natural frequency of P-FGM square plate with SCSC edge condition are presented in Fig. 3.10 (a-c). It is illustrated from these figures that as the value of  $E_{rat} = \rho_{rat}$  increases (0 to 40), the natural plate frequency is decreased. At the low value of  $E_{rat} = \rho_{rat}$  (say, less than 10), the decrease in natural frequency is noticeable, and beyond this, it is not significant for a particularly lower value of material gradient index.

The variations of density ratio ( $\rho_{rat}$ ) for different material gradient index (p) with different combinations of elastic modulus ( $K_{w}$ ,  $K_{P}$ ) on the natural frequency of square P-FGM plate for SSSS edge condition and fixed value of  $E_{rat} = 2$ , are shown in Fig. 3.11(a-c). The pattern of natural frequency variation for all boundary condition (Levy-type) are same, therefore, SSSS boundary condition in Levy type solution is considered and as shown in Fig. 3.11 (a-c). The impact of the Pasternak modulus is considerable on natural frequency than the Winkler modulus for different values of material gradient index. The effect of low-density ratio ( $\rho_{rat} < 10$ ) with all values of material gradient index does not show a significant impact on the natural frequency with different combinations of elastic modulus ( $K_{w}$ ,  $K_{P}$ ).

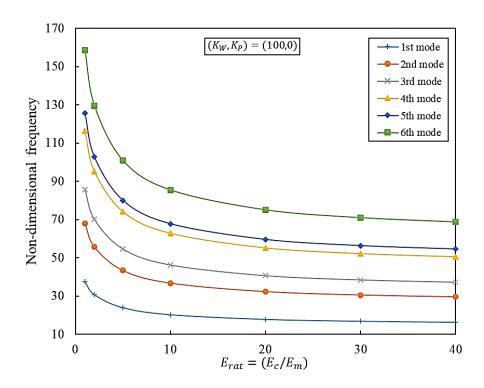
The impact of Winkler-Pasternak modulus  $(K_{w_i}K_P)$  for different values of aspect ratio (a/b)

on the natural frequency of rectangular P-FGM plate with SSSS edge condition under the fixed value of material gradient index (p = 1), and  $E_{rat} = 1$ ,  $\rho_{rat} = 1$  are shown in Fig.3.12(a-c).

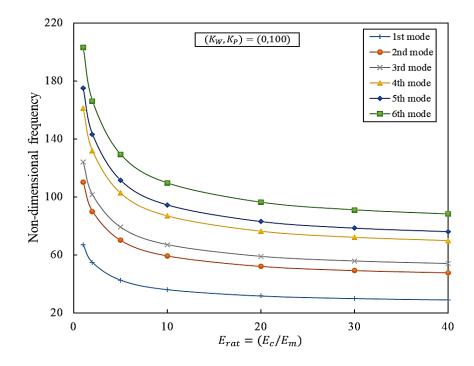
It is obtained from these figures results that as the aspect ratio (a/b) increases, the natural frequency of the P-FGM plate is increased. This is because of increases the flexural stiffness of the plate. The reported results show that the Pasternak modulus's effect is more significant than the Winker modulus for all aspect ratio values. It is also noticed that the natural frequency does not change for a high value of aspect ratio for Winkler modulus but increases in the case of Pasternak modulus.



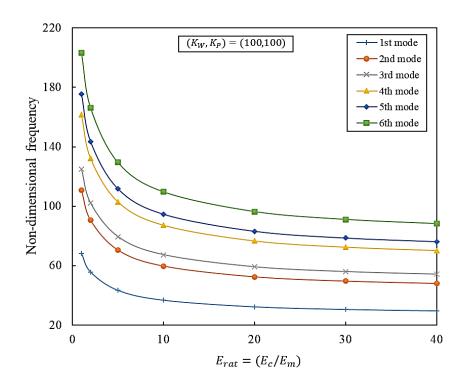
**Fig. 3.8 (a):** Effect of  $E_{rat}$  on frequency parameter  $(\omega^*)$  for square P-FGM plate for constant  $(K_W, K_p = 0.0)$ , p = 2,  $\rho_{rat} = 2$ , and h/a = 0.01.



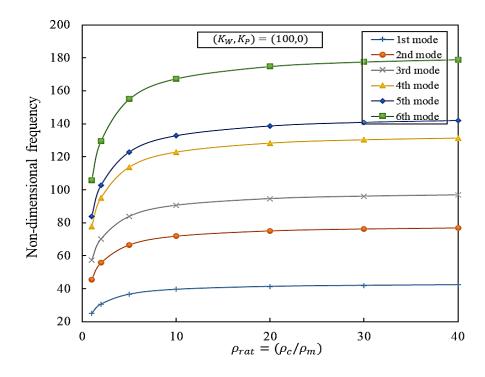
**Fig. 3.8 (b):** Effect of  $E_{rat}$  on frequency parameter ( $\omega^*$ ) for square P-FGM plate for constant  $(K_W, K_p = 100,0)$ , p = 2,  $\rho_{rat} = 2$ , and h/a = 0.01.



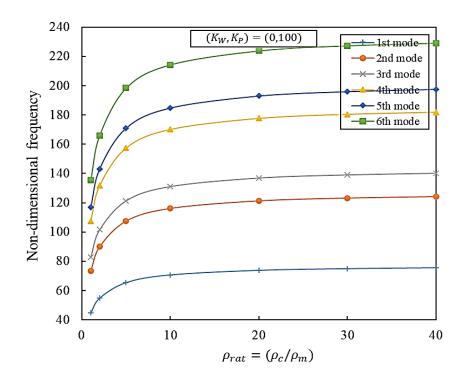
**Fig. 3.8 (c):** Effect of  $E_{rat}$  on frequency parameter ( $\omega^*$ ) for square P-FGM plate for constant  $(K_W, K_p = 0.100)$ , p = 2,  $\rho_{rat} = 2$ , and h/a = 0.01.



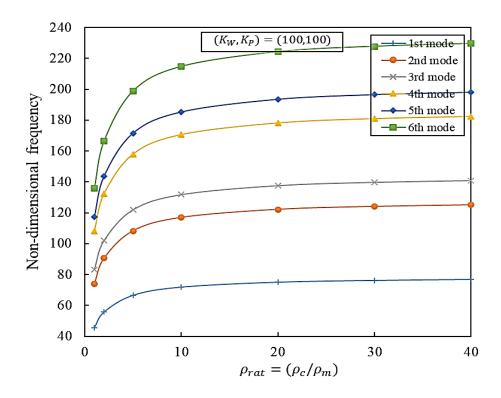
**Fig. 3.8 (d):** Effect of  $E_{rat}$  on frequency parameter  $(\omega^*)$  for square P-FGM plate for constant  $(K_W, K_p = 100, 100)$ , p = 2,  $\rho_{rat} = 2$ , and h/a = 0.01.



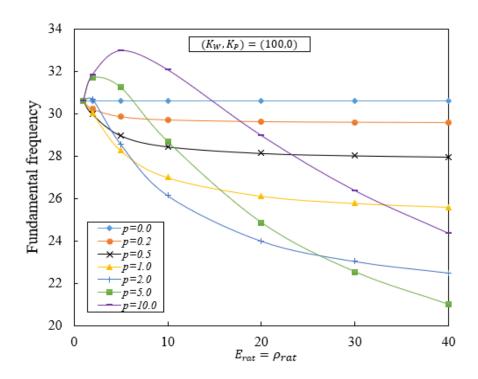
**Fig. 3.9 (a):** Effect of  $\rho_{rat}$  on frequency parameter ( $\omega^*$ ) for square P-FGM plate for constant  $(K_W, K_p = 100,0)$ , p = 2,  $E_{rat} = 2$ , and h/a = 0.01.



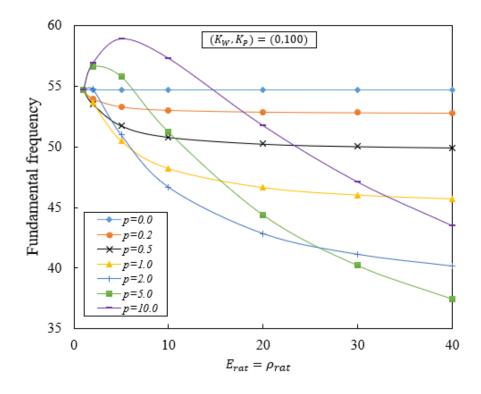
**Fig. 3.9 (b):** Effect of  $\rho_{rat}$  on frequency parameter ( $\omega^*$ ) for square P-FGM plate for constant  $(K_W, K_p = 0.100)$  p = 2,  $E_{rat} = 2$  and h/a = 0.01.



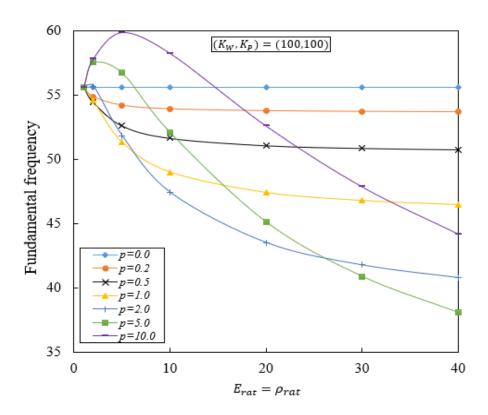
**Fig. 3.9 (c):** Effect of  $\rho_{rat}$  on frequency parameter  $(\omega^*)$  for square P-FGM plate for constant  $(K_W, K_p = 100,100)$  p = 2,  $E_{rat} = 2$  and h/a = 0.01.



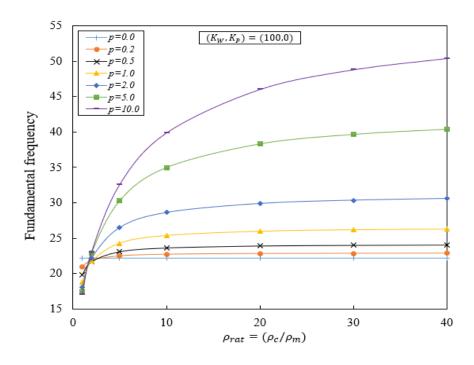
**Fig. 3.10** (a): Effect of p on frequency parameter ( $\omega^*$ ) for square P-FGM plate for constant  $(K_W, K_p = 100,0)$ ,  $(\rho_{rat} = E_{rat})$  and h/a = 0.01.



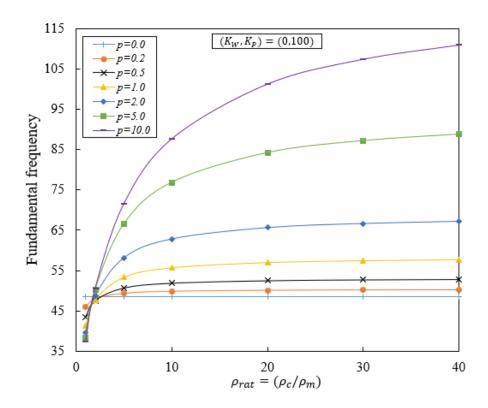
**Fig. 3.10 (b):** Effect of p on frequency parameter  $(\omega^*)$  for square P-FGM plate for constant  $(K_W, K_p = 0.100)$ ,  $(\rho_{rat} = E_{rat})$  and h/a = 0.01.



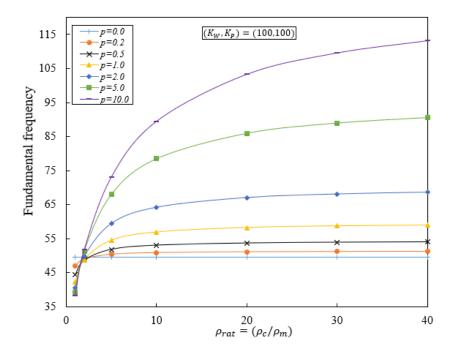
**Fig. 3.10 (c):** Effect of p on frequency parameter  $(\omega^*)$  for square P-FGM plate for constant  $(K_W, K_p = 100,100)$ ,  $(\rho_{rat} = E_{rat})$  and h/a = 0.01.



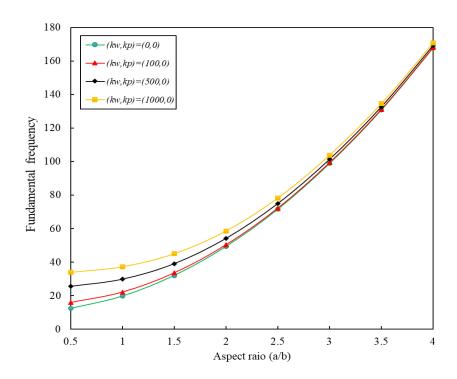
**Fig. 3.11 (a):** Effect of p and  $\rho_{rat}$  on frequency parameter ( $\omega^*$ ) for square P-FGM plate under SSSS boundary condition for constant  $(K_W, K_p = 100,0), E_{rat} = 2$  and h/a = 0.01.



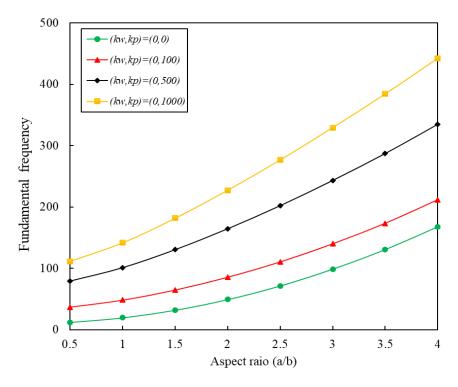
**Fig. 3.11 (b):** Effect of p and  $\rho_{rat}$  on frequency parameter ( $\omega^*$ ) for square P-FGM plate for SSSS boundary condition for constant ( $K_W$ ,  $K_p = 0.100$ ),  $E_{rat} = 2$ , and h/a = 0.01.



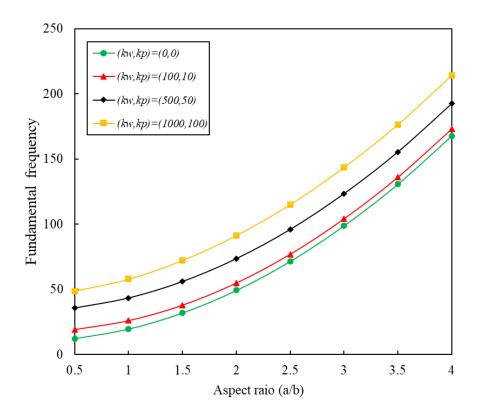
**Fig. 3.11 (c):** Effect of p and  $\rho_{rat}$  on frequency parameter ( $\omega^*$ ) for square P-FGM plate for SSSS boundary condition for constant ( $K_W, K_p = 100,100$ ),  $E_{rat} = 2$ , and h/a = 0.01.



**Fig.3.12** (a): Effect of aspect ratio on frequency parameter ( $\omega^*$ ) under different combinations of  $K_W$  and fixed  $K_p = 0$  for SSSS boundary conditions for p = 1,  $E_{rat} = 2$ ,  $\rho_{rat} = 2$  and h/a = 0.01.



**Fig. 3.12 (b):** Effect of aspect ratio on frequency parameter  $(\omega^*)$  under different combinations of  $K_p$  and fixed  $K_W = 0$  for SSSS boundary conditions for p = 1,  $E_{rat} = 2$ ,  $\rho_{rat} = 2$  and  $h/\alpha = 0.01$ .



**Fig. 3.12 (c):** Effect of aspect ratio on natural frequency parameter ( $\omega^*$ ) under different combinations of  $K_W$  and  $K_p$  for SSSS boundary conditions for p=1,  $E_{rat}=2$ ,  $\rho_{rat}=2$  and h/a=0.01.

#### 3.4. Summary

In this chapter of the thesis the dynamic stiffness method for thin rectangular P-FGM plate resting on Winkler and Pasternak elastic foundations was developed. The concept of a physical neutral surface instead of mid-surface with classical plate theory is implemented for the mathematical formulation of a thin P-FGM plate. The application of Hamilton's principle is applied to obtain the governing partial differential equation of motion under the consideration of elastic foundation. A levy-type solution is used to implement the force and displacement boundary condition at the opposite edge and develop the dynamic stiffness matrix of the P-FGM plate. The Wittrick-Williams algorithm is implemented as a solution method for solving the transcendental nature of the dynamic stiffness matrix and extracting the natural frequencies or eigenvalue of the P-FGM plate supported on an elastic foundation.

The methodology of DSM formulation has been employed in a MATLAB software program to extract the natural frequencies of the P-FGM plates supported on an elastic foundation. These DSM natural frequency results of P-FGM plate are compared with available published literature and noticed that these results are very well agreement. This present work contributes a new set of natural frequencies results incorporating the elastic foundation for square and rectangular P-FGM plates.

The impact of different material parameters, boundary conditions, and elastic foundation on natural frequencies is also reported in tables and graphs. The present work shows that the effect of the Pasternak modulus on the natural frequency of the P-FGM plate is more than the Winkler modulus for different material property variations (material gradient index, boundary conditions, aspect ratio). It is observed that the developed natural frequencies results for the P-FGM plate supported on an elastic foundation are satisfactorily accurate and can be referred further as a benchmark standard solution for future comparison.

### **CHAPTER 4**

# Free Vibration Analysis of S-FGM Plate Resting on Elastic Foundation

#### 4.1 Introduction

As described earlier in Chapter 1, The mechanical properties variation of the FGM plates in the transverse/thickness direction are modeled according to different mathematical expressions such as power law, sigmoidal law, and exponential law. The material property variation of FGM plates by power law has relatively well reported in the literature. It is noticed from the literature that the natural vibration characterization of the FGM plates resting on elastic foundations under sigmoid-law material property variation in the transverse direction has relatively not well attention. Therefore, in this present chapter of the thesis, sigmoid law is chosen as material properties variation method as it ensures the smooth distribution of stresses in FGM plate. Here, similar to the P-FGM plate case, the development of the dynamic stiffness method to extract the free vibration frequency of sigmoid-law FGM plate with the Winkler and Pasternak elastic foundation. The variation of material properties are continuously vary along the thickness direction of the FGM plate using two power-law material property variations in terms of volume fraction of the constituent's material. Hamilton's principle is applied to obtain the governing partial differential equation of motion based on the CPT under the physical neutral surface of the FGM plate. The Wittrick-Williams algorithm is used as a solution technique to solve the transcendental nature of the dynamic stiffness matrix and carry out the free vibration frequencies of the FGM plate with the desired accuracy.

The variation of free vibration frequencies with the change of parametric numerical values (aspect ratio, sigmoid volume fraction index, boundary conditions and elastic foundation parameters, density ratio, modulus ratio) of the S-FGM plates are also reported. The DSM results are compared and validated with the reported published literature. A new set of natural frequency results for S-FGM plate embedded on Winkler-Pasternak elastic foundation are generated.

#### 4.2 Contributions and relevant scope

This present study analyses the response of free vibration in the transverse direction of the thin S-FGM plates embedded on the Winkler-Pasternak elastic medium, where sigmoidal law is used to describe the material property variations in terms of volume fraction of the given constituents. The displacement components of the S-FGM plate are described according to CPT. Therefore, the shear deformation effect of the plate is neglected, and therefore this research is valid only for the thin S-FGM plates. The boundary conditions of levy type are applied where two contrary edges of the S-FGM plate are simply supported, and the remaining are free, simply-supported or clamped. These are the scope and limitations of the present study where the contribution of the work may be compiled as follows.

- The DS matrix is formulated to evaluate the response of free vibration behavior of S-FGM plate embedded on Winkler- Pasternak elastic foundation.
- Wittrick-Williams algorithm is implemented to extract the accurate natural frequencies of the S-FGM plate.
- Instead of a geometrical mid surface, a physical neutral surface is applied to model the S-FGM plate.
- The natural frequency of the S-FGM plates are compared and validated with the published results to prove that the present obtained results are highly accurate. It can be used further to examine the accuracy of different methods.
- A new set of frequencies of S-FGM plates embedded on Winkler- Pasternak elastic foundation are obtained and reported for different sigmoid volume fraction index, Winkler and Pasternak elastic modulus, modulus ratio, density ratio and aspect ratio.

#### 4.3 Theoretical formulation of DSM

#### 4.3.1 Geometry and material properties description of S-FGM plate

The Cartesian coordinate system and material geometry description of the sigmoid-law (S-FGM) functionally graded plate are deliberated to be similar to that of power-law (P-FGM) functionally graded plate as described in the previous Section 3.2.1. The Schematic representation of S-FGM plate can be redraw with different orientation and shown in Fig. 4.1, where the geometric descriptions of the plate are described as length (a), width (b), and the plate thickness (h). The S-FGM plate is the combination of metal and ceramic constituent, where the upper part of the given plate is highly rich in ceramic, and the lower part is highly

rich in metallic. The material properties vary in the transverse direction in S-FGM by double power law functions [37-39]in terms of volume fraction of the plate as defined by Eq. (4.1), which ensure that the distribution of stresses within the material are smooth.

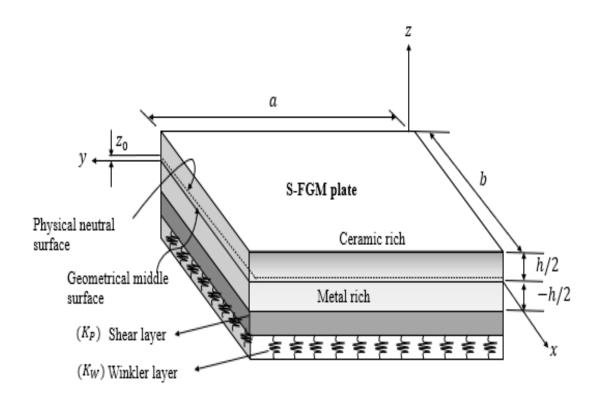


Fig. 4.1: Schematic representation of S-FGM plate on the Winkler-Pasternak elastic foundation

The expression of volume fraction of sigmoid function can be given by

$$V_1(z) = 1 - \frac{1}{2} \left( 1 - \frac{2z}{h} \right)^k \text{ for } 0 \le z \le h/2$$

$$V_2(z) = \frac{1}{2} \left( 1 + \frac{2z}{h} \right)^k \text{ for } -h/2 \le z \le 0$$
(4.1)

where  $V_1(z)$ ,  $V_2(z)$  represent the volume fractions of the S-FGM plate.

The non-negative material parameter is represented by (k) and known as sigmoid power law index (or volume fraction index), which defines the material volume fraction. To determine the material properties, a rule of mixture is applied and expressed by Eq. (4.2).

$$P_1(z) = V_1(z)P_c + [1 - V_1(z)]P_m \quad \text{for } 0 \le z \le h/2$$

$$P_2(z) = V_2(z)P_c + [1 - V_2(z)]P_m \quad \text{for } -h/2 \le z \le 0$$
(4.2)

where  $P_m$  and  $P_c$  are the material properties of metal and ceramic, respectively.

The two halves variation of material properties of S-FGM are represented by Young's modulus  $E_1(z)$ ,  $E_2(z)$  and density,  $\rho_1(z)$ ,  $\rho_2(z)$ , respectively and described by Eq. (4.2). The detail description of this mathematical model is explained in the Chapter 1 and material properties are considered as same as that of P-FGM plate. Refer Section 1.2.2 of Chapter 1 for better understanding of Young's modulus variation of the S-FGM plate.

#### 4.3.2. Elastic foundation models

As same as Section 3.3.2 of Chapter 3, this present work implemented Winkler and Pasternak elastic foundation for analyzing the natural vibration of S-FGM plate and the detail description of Winkler-Pasternak model in the FGM plate can be explained in Section 3.2.2 of Chapter 3. Here, advantage of this elastic foundation model is that it considers both normal pressure and transverse shear deformation of the surrounding elastic medium. Considering this advantage the free vibration behavior of the foundation, structure interaction is widely described by this model.

Reaction force or normal force [48] of the foundation can be expressed as

$$q_{\mathrm{Winkler}} = k_W w$$

where q represents the reaction force or normal force of the foundation and w represent the vertical displacement, 
$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$
,  $k_W$  and  $k_P$  represent the Winkler and

(4.3)

Pasternak elastic modulus, respectively.

 $q_{\text{Pasternak}} = k_W w - k_P \nabla^2 w$ 

#### 4.3.3. Discussion of the physical neutral surface

Here, the classical plate theory (CPT) along with the physical neutral surface (PNS) to determine the displacement components of the given plate same as determined in Chapter 3. The detail description of PNS of FGM plate along the transverse direction of the material property is explained in the Section 3.2.3 of Chapter 3. For determining the physical neutral

surface  $(z_0)$ , the total axial forces of the plate in the directions of x or y should be equal to zero. Therefore,

$$\sum F_x = \int_{-h/2 - z_0}^{h/2 - z_0} \sigma_{xx} dA = 0 \tag{4.4}$$

which gives to

$$z_0 = \frac{\int_{-h/2}^{h/2} E(z)zdz}{\int_{-h/2}^{h/2} E(z)dz}$$
(4.5)

The mathematic expression of the PNS  $(z_0)$  of the S-FGM plate can be expressed as

$$(z_0) = \frac{\int_0^{h/2} E_1(z)zdz + \int_{-h/2}^0 E_2(z)zdz}{\int_0^{h/2} E_1(z)dz + \int_{-h/2}^0 E_2(z)dz}$$

$$= \frac{h}{2(E_c + E_m)} [(E_c - E_m)/2 + (E_m - E_c)/(k+1)(k+2)]$$

$$= \frac{h}{2(E_{ra} + 1)} [(E_{rat} - 1)/2 + (1 - E_{rat})/(k+1)(k+2)]$$
(4.6)

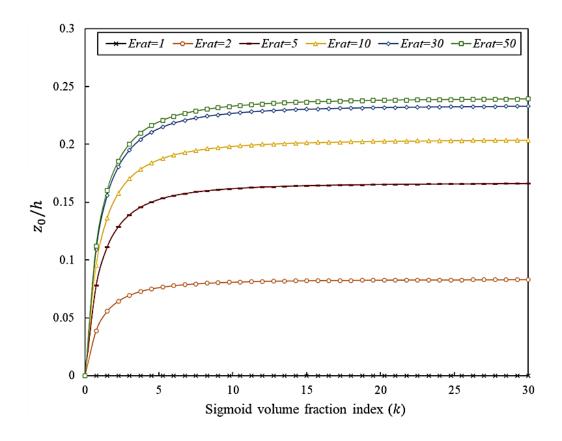


Fig. 4.2: Shifting of non-dimensional  $(z_0/h)$  with sigmoid volume fraction index (k)

It is observed from Eq. (4.6) that the value of  $z_0$  is the function of Young's modulus ratio,  $E_{rat} = E_c/E_m$  and sigmoid law index (k) of the plate. The variation of  $z_0$  with different values of  $E_{rat}$  corresponding to different sigmoid volume fraction index is shown in Fig. (4.2). At  $z_0 = 0$  for  $E_{rat} = 1$ , the S-FGM plate is considered a homogeneous isotropic plate, and the PNS coincides with the geometrical mid surface. As the  $E_{rat}$  value increases,  $z_0$  also increases in Eq.4.6, and for the larger value of k, it converges to an asymptotic value expressed as  $\frac{z_0}{h} = (E_{rat} - 1)/4(E_{rat} + 1)$ . As the value of  $E_{rat}$  increases, the PNS of the S-FGM plate is shifted away from the mid surface of the plate and moves towards the higher ceramic rich side of the plate. This phenomenon occurs because of the high stiffness value of ceramic constituents present at the upper side compared to the metal constituents present at the lower side of the S-FGM plate.

#### 4.3.4. The free vibration equation of motion of the S-FGM plate

The formulation of free vibration equation of motion of S-FGM plate follow the same procedure of P-FGM plate as explained in Section 3.2.4 of Chapter 3, and can be used to obtain the governing differential equation for free vibration of the plate. It is noticed that the same natural boundary condition and governing differential equation are obtained and describe in Eq. 3.15 and 3.16 of the previous Chapter 3. For continuation of present thesis work, these equations can be rewritten as

$$D_{SFGM}\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + I_0\frac{\partial^2 w}{\partial t^2} + k_W\frac{\partial^2 w}{\partial t^2} + k_P\left(\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\partial^4 w}{\partial y^2 \partial t^2}\right) = 0 \tag{4.7}$$

The natural boundary conditions on plate element are given as shown in Fig. (3.3) of section 3.2.4 of Chapter 3.

$$V_{x} = \left[ -D_{SFGM} \left( \frac{\partial^{3} w}{\partial x^{3}} + (2 - v) \frac{\partial^{3} w}{\partial x \partial y^{2}} \right) \right] \delta w$$

$$M_{xx} = -D_{SFGM} \left( \frac{\partial^{2} w}{\partial x^{2}} + v \frac{\partial^{2} w}{\partial y^{2}} \right) \delta \phi_{y}$$

$$(4.8)$$

where  $D_{SFGM}$  represents the S-FGM plates flexure rigidity,  $V_x$ , and  $M_{xx}$  represents the shear force and bending moment of the FGM plate. The corresponding mathematical expressions can be expressed in Eq. 4.9 and 4.10. A non-dimensional parameter  $D_{SFGM}/D_C$  is introduced, where  $D_C = E_c h^3/12(1-v^2)$ .

$$(D_{SFGM}) = \int_{-h/2-z_0}^{h/2-z_0} (z_{ns}^2) Q_{11}(z_{ns}) dz_{ns} = \int_{-h/2}^{h/2} Q_{11}(z) (z - z_0)^2 dz$$

$$= \frac{h^3}{2(1-v^2)} \left[ \frac{E_c + E_m}{12} - \left( \frac{z_0}{h} \right) \left\{ \frac{(E_m - E_c)}{(k+1)(k+2)} + \frac{(E_c - E_m)}{2} \right\} \right.$$

$$+ (E_c + E_m) \left( \frac{z_0}{h} \right)^2 \right]$$

$$= \frac{6D_c}{E_{rat}} \left[ \frac{E_{rat} + 1}{12} - \left( \frac{z_0}{h} \right) \left\{ \frac{(1 - E_{rat})}{(k+1)(k+2)} + \frac{(E_{rat} - 1)}{2} \right\} \right.$$

$$+ (E_{rat} + 1) \left( \frac{z_0}{h} \right)^2 \right]$$

$$(4.9)$$

$$(I_0) = \int_{-\frac{h}{2} - z_0}^{\frac{h}{2} - z_0} \rho(z_{ns}) dz_{ns} = \int_{-h/2}^{h/2} \rho(z) dz = \rho_c h\left(\frac{\rho_{rat} + 1}{2\rho_{rat}}\right)$$
(4.10)

where  $I_o$  represents the inertia in the transverse direction of the FGM plate. When  $E_{rat} = 1$ , the S-FGM plate is converted into a pure ceramic-rich plate, and the non-dimensional parameter  $(D_{SFGM}/D_C)$  becomes one. This implies,  $D_{SFGM} = D_C$ . The non-dimensional variation of  $D_{SFGM}/D_C$  with six different values of  $E_{rat}$  for different sigmoid volume fraction index is presented in Fig. 4.3. The non-dimensional parameter  $D_{SFGM}/D_C$  decreases with an increase in  $E_{rat}$  and k. With an increase in the value of  $E_{rat}$  and k, the volume fraction of metal constituents increases which has lesser stiffness and Young's modulus than the ceramic constituents, as shown in Fig. 4.3.

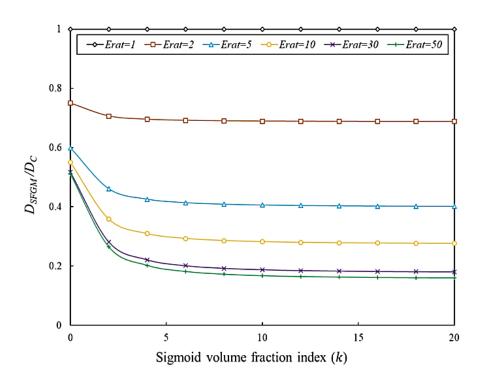


Fig. 4.3: The non-dimensional parameter  $D_{SFGM}/D_C$  with different k and  $E_{rat}$ .

The natural boundary conditions of the plate in Eq. (4.9) indicate that the shear force  $(V_x)$  is the function of displacement component (w), and the moment is the function of rotation  $\phi_y = \frac{\partial w}{\partial x}$ . In the formulation of the DS matrix, Eq. (4.8) and Eq. (4.9) are used as key elements, and follow the similar methodology in section 3.3.3 of Chapter 3 here to obtain the DS matrix for S-FGM plate.

#### 4.3.5 Natural frequency computation using DSM

The formulation of dynamic stiffness matrix for S-FGM plate resting on elastic foundation follows the same procedures of P-FGM plate as explained in earlier Section 3.2.6 of Chapter 3. A similar type of 4x4 square DS matrix K is obtained with six independent terms which is dependent (or function) on natural frequency of S-FGM plates. Here, the DS matrixes with six independent terms are dependent on material property variation of sigmoid law. But in previously these six independent terms are dependent on material property variation of power law, this is the only difference between the both DS matrix in P-FGM and S-FGM plate with the corresponding changes of  $D_{FGM}$  and  $I_0$  terms.

As mentioned earlier that the block assembly formulation of DSM is the same as FEM, but DSM differs in the discretization technique of the structures. In FEM, the assumed shape function is used to discretize the structural element and to obtain separate stiffness and mass matrix, whereas DSM uses eigenvalue-dependent exact shape function to obtain a matrix of a single element that contains both stiffness and mass properties called dynamic stiffness matrix. Consequently, dealing with same penalty method under applied all possible boundary conditions (Levy-type) of the FGM plate. The description of assembly process and penalty method is explained in pervious Section 3.2.7 of Chapter 3.

After completing the assembly process, the obtained global DS matrix is used for estimating the accurate natural frequency results for the S-FGM plates by using the application of Wittrick and Williams algorithm. This algorithm is computationally efficient, accurate, robust and reliable with used the Sturm sequence property of the global matrix and ensure that there is no natural frequency is missed in the given structures. Therefore, the W-W algorithm is well suitable algorithm for DSM and the detail procedure of the algorithm has been explained in the Section 3.2.8 of Chapter 3. In the next sections, the natural frequency DSM results for S-FGM rectangular plates are explained in derails along with proper interpretation and explanations of the obtained results.

#### 4.4 Results and discussion

In this section, the natural frequencies and mode shapes of S-FGM plate are extracted using a software program, MATLAB. Same as earlier, the material property of ceramic and metal  $(Al_2O_3/Al)$  are used in Table 3.1 of Chapter 3 is implemented in S-FGM plate with similar symbol and notations such as the symbol 'C' 'F' and 'S' represent clamped, free and simply supported sides of the S-FGM plate, respectively. For example, the boundary condition SFSC stands for simply supported (at y = 0 and y = a), free (at x = 0), and clamped (at x = a). The symbols, x = a0 and x = a1 frequency mode shape, where x = a2 represent the number of half sine waves in the x = a3 direction and x = a4 represent the x = a5 lower frequency for a corresponding value of x = a4 replied in this present work of the thesis.

The comparative study of the natural frequency obtained by DSM with those available in the published literature is presented in this section. Finally, the effect of design parameters (sigmoid volume fraction index, Winkler and Pasternak elastic modulus, modulus ratio, density ratio and aspect ratio.) on the natural frequency in the form of tables and graphs are also highlighted.

#### 4.4.1 Comparative study

For effective comparison of natural frequency obtained by DSM, the following expressions for non-dimensional natural frequency parameters are considered [101, 142].

$$\overline{\omega} = \omega h \sqrt{\rho_c/G_c} , \quad \widetilde{\omega} = \omega h \sqrt{\rho_c/E_c}, \quad \omega^* = \omega \alpha^2 \sqrt{\frac{\rho_c h}{D_c}}, \quad K_p = \frac{k_p \alpha^4}{D_c}, \quad K_w = \frac{k_w \alpha^2}{D_c} \quad (4.11)$$

where  $G_c = E_c/(2(1 + \nu))$  represents the shear modulus of the ceramic constituents of the plate.

For k=0, the S-FGM plate reduces to the homogeneous isotropic plate. Therefore, the natural frequency parameters obtained by DSM of square (a/b=1) and rectangular  $(a/b=\sqrt{2})$ , homogeneous isotropic plate having simply supported at all edges with h/a=0.1 are validated with those in published literature and are shown in Table 4.1.

**Table 4.1:** Validation of natural frequencies parameters  $(\overline{\omega} = \omega h \sqrt{\rho_c/G_c})$  of SSSS homogeneous isotropic plate.

			Ref.	Ref.	Ref.				Ref.	Ref.	
a/b	m n	Present	[11]	[103]	[268]	a/b	m n	Present	[11]	[103]	Ref. [268]
1	1 1	0.0963	0.0963	0.0963	0.0963	$\sqrt{2}$	1 1	0.0722	0.0722	0.0722	0.0722
	1 2	0.2408	0.2408	0.2408	0.2408		2 1	0.1445	0.1445	0.1445	0.1445
	2 2	0.3853	0.3853	0.3853	0.3853		1 2	0.2167	0.2167	0.2167	0.2167
	13	0.4816	0.4816	0.4816	0.4816		3 1	0.2649	0.2649	0.2649	0.2649
	23	0.6261	0.6261	0.6261	0.6261		2 2	0.2890	0.2890	0.2890	0.2890
	1 4	0.8187	0.8187	0.8187	0.8187		3 2	0.4094	0.4094	0.4094	0.4094
	3 3	0.8669	0.8669	0.8669	0.8669		4 1	0.4334	0.4334	0.4334	0.4334
	2 4	0.9632	0.9632	0.9632	0.9632		1 3	0.4575	0.4575	0.4575	0.4575
	3 4	1.2040	1.2040	1.2040	1.2040		2 3	0.5298	0.5298	0.5298	0.5298
	1 5	1.2521	1.2521	1.2521	1.2521		4 2	0.5779	0.5779	0.5779	0.5779
	2 5	1.3966	1.3966	1.3966	1.3966		3 3	0.6501	0.6501	0.6501	0.6501
	4 4	1.5411	1.5411	1.5411	1.5411		5 1	0.6501	0.6501	0.6501	0.6501
	3 5	1.6374	1.6374	1.6374	1.6374		5 2	0.7946	0.7946	0.7946	0.7946

The results obtained by DSM method are in excellent agreement.

The fundamental frequencies of the square and rectangular S-FGM plate for different sigmoid volume fraction index, k obtained by DSM are compared with those published in the literature in Tables 4.2 and 4.3.

**Table 4.2:** Comparison of natural frequency parameters ( $\widetilde{\omega} = \omega h \sqrt{\rho_c/E_c}$ ) of square S-FGM plate under SSSS boundary conditions for different h/a and sigmoid volume fraction index, k.

(a/h)	Source	k = 1	k = 2	k = 5	k = 10	
5	Ref. [269]	0.1631	-	-	-	
	Ref. [270]	0.1631	-	-	-	
	Ref. [43]	0.1631	-	-	-	
	Ref. [271]	0.1631	0.1554	0.1484	0.1462	

	Ref. [260]	0.1631	0.1554	0.1484	0.1462
	Ref. [181]	0.1631	0.1554	0.1484	0.1462
	Present	0.1631	0.1554	0.1484	0.1462
10	Ref. [269]	0.0442	-	-	-
	Ref. [270]	0.0442	-	-	-
	Ref. [43]	0.0442	-	-	-
	Ref. [271]	0.0442	0.0419	0.0398	0.0392
	Ref. [260]	0.0442	0.0419	0.0398	0.0392
	Ref. [181]	0.0442	0.0419	0.0398	0.0392
	Present	0.0442	0.0419	0.0398	0.0392
20	Ref. [269]	0.0113	-	-	-
	Ref. [270]	0.0113	-	-	-
	Ref. [43]	0.0113	-	-	-
	Ref. [271]	0.0113	0.0107	0.0102	0.0100
	Ref. [260]	0.0113	0.0107	0.0102	0.0100
	Ref. [181]	0.0113	0.0107	0.0102	0.0100
	Present	0.0113	0.0107	0.0102	0.0100

**Table 4.3:** Comparison of natural frequency parameters  $(\overline{\omega} = \omega h \sqrt{\rho_c/G_c})$  of S-FGM rectangular plate with SSSS boundary condition for h/a = 0.05.

( <i>a</i> / <i>b</i> )	m n	Source	k = 1	k = 2	k = 5	k = 10
0.5	1 1	Ref. [269]	-	-	-	-
		Ref. [270]	-	-	-	-
		Ref. [103]	0.0283	0.0268	0.0254	0.0249
		Ref. [43]	-	-	-	-
		Ref. [160]	0.0284	0.0268	0.0254	0.025
		Present	0.0284	0.0268	0.0254	0.025
	12	Ref. [133]	-	-	-	-
		Ref. [134]	-	-	-	-
		Ref. [103]	0.0452	0.0427	0.0405	0.0398
		Ref. [43]	-	-	-	-
		Ref. [124]	-	-	-	-

		Present	0.0452	0.0427	0.0406	0.0399
	13	Ref. [133]	-	-	-	-
		Ref. [134]	-	-	-	-
		Ref. [103]	0.073	0.069	0.0654	0.0643
		Ref. [43]	-	-	-	-
		Ref. [124]	0.0731	0.0692	0.0654	0.0643
		Present	0.0731	0.0692	0.0654	0.0643
	2 1	Ref. [133]	-	-	-	-
		Ref. [134]	-	-	-	-
		Ref. [103]	0.0951	0.0899	0.0852	0.0837
		Ref. [43]	-	-	-	-
		Ref. [124]	0.0953	0.0902	0.0856	0.0842
		Present	0.0951	0.0899	0.0852	0.0837
2	11	Ref. [133]	2.7937	-	-	-
		Ref. [134]	2.7937	-	-	-
		Ref. [103]	-	-	-	-
		Ref. [43]	2.7937	2.6456	2.5124	2.4716
		Ref. [124]	2.7937	2.6456	2.5124	2.4716
		Present	2.7937	2.6456	2.5124	2.4716
	1 2	Ref. [133]	4.4192	-	-	-
		Ref. [134]	4.4194	-	-	-
		Ref. [103]	-	-	-	-
		Ref. [43]	4.4192	4.1884	3.9804	3.9166
		Ref. [124]	4.4192	4.1884	3.9804	3.9166
		Present	4.4192	4.1884	3.9804	3.9166
	13	Ref. [133]	7.0512	-	-	-
		Ref. [134]	7.0519	-	-	-
		Ref. [103]	-	-	-	-
		Ref. [43]	7.0515	6.6913	6.3664	6.2666
		Ref. [124]	7.0515	6.6913	6.3664	6.2666
		Present	7.0515	6.6913	6.3664	6.2666
	2 1	Ref. [133]	9.5261	-	-	-
		Ref. [134]	9.5263	-	-	-

Ref. [103]	-	-	-	-
Ref. [43]	9.5261	9.0195	8.5638	8.424
Ref. [124]	9.5261	9.0195	8.5638	8.424
Present	9.5261	9.0195	8.5638	8.424

Tables (4.2-4.3) show the natural frequencies of S-FGM plate under SSSS boundary conditions for different values of k. The results obtained by DSM agree very well with those published in the literature.

Table 4.4 represents the comparison of the fundamental natural frequency obtained by DSM for SSSS boundary condition for four different sigmoid volume fraction indexes with different combinations of Winkler-Pasternak elastic modulus. The results are in excellent agreement.

**Table 4.4:** Comparison of natural frequency parameters ( $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$ ) of SSSS square S-FGM plate for different values of Winkler-Pasternak parameters with different sigmoid volume fraction index for h/a = 0.01.

$K_w$	$K_P$	Source	k = 1	k = 2	k = 5	k = 10
0	0	Ref. [272]	8.9549	8.4686	8.0319	7.8981
		Ref. [189]	8.9549	8.4686	8.0319	7.8981
		Present	8.9549	8.4686	8.0319	7.8981
0	$10^{2}$	Ref. [272]	15.1789	14.8972	14.6533	14.5804
		Ref. [189]	15.1789	14.8972	14.6533	14.5804
		Present	15.1789	14.8972	14.6533	14.5804
$10^{2}$	0	Ref. [272]	9.3702	8.9066	8.4924	8.3660
		Ref. [189]	9.3702	8.9066	8.4924	8.3660
		Present	9.3702	8.9066	8.4924	8.3660
$10^{2}$	$10^{2}$	Ref. [272]	15.4276	15.1504	14.9107	14.8390
		Ref. [189]	15.4276	15.1504	14.9107	14.8390
		Present	15.4276	15.1504	14.9107	14.8390

#### 4.4.2. Natural frequency of S-FGM plate with Winkler-Pasternak elastic foundation

In this section, a new set of the natural frequencies of S-FGM plates obtained by DSM for two geometric configurations, i.e., one square and another rectangular with Winkler-Pasternak elastic foundation is reported.

#### 4.4.2.1. Natural frequency of square S-FGM plate with Winkler-Pasternak elastic foundation

The first six non-dimensional natural frequencies of a square plate (k=0) for all levy type boundary conditions with four different values of Winkler modulus ( $K_w=0,10,100,1000$ ) are shown in Table 4.5. The natural frequency of a given mode increases with an increase in the Winkler modulus ( $K_w$ ) from 0 to 1000. This is because of increase in the stiffness of the plate structure. Table 4.5 shows that the non-dimensional natural frequency is maximum for boundary condition of SCSC and minimum for boundary condition of SFSF. The reason is that the more constraint added at the boundary of the plate, the higher the stiffness of the plate, which causes an increase in the natural frequencies. In Table 4.6, the natural frequency of the S-FGM plate increases as Pasternak modulus increases from 0 to 1000 due to increase in the stiffness of the plate. It is observed that the influence of Pasternak modulus ( $K_w=0$ ) is more than the Winkler modulus ( $K_P=0$ ) as shown in Tables (4.5-4.6).

Table 4.7 presents the fundamental natural frequencies of S-FGM plate for different Winkler-Pasternak modulus, different values of sigmoid volume fraction index for levy type boundary conditions. It is observed that the natural frequencies decrease with an increase in the sigmoid volume fraction indices (k) for a given Levy type's boundary condition and for a given combination of elastic foundation. As k increases, stiffness of the S-FGM plate decrease, which means more metal, is introduced in the given structure. The representative mode shapes for square S-FGM plates for Winkler-Pasternak modulus  $(K_w, K_P) = (100,100)$ , sigmoid volume fraction index, k = 1 and for SSSS boundary condition, are shown in Fig. 4.7. The obtained mode shapes are almost of similar pattern as observed in isotropic plates.

**Table 4.5:** Non-dimensional fundamental frequencies parameters ( $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$ ) for square isotropic S-FGM plate with levy type boundary conditions for different value of Winkler parameters for  $K_p = 0$  and h/a = 0.01.

		Mode no.	Mode no.					
BCs	$K_w$	1	2	3	4	5	6	

SCSC	0	24.0756	45.5245	57.6525	78.6574	85.0033	107.3562
	$10^{2}$	25.4714	46.2778	58.2492	79.0957	85.4091	107.6778
	$10^{3}$	35.6539	52.5742	63.3670	82.9370	88.9782	110.5301
SCSS	0	19.6643	42.9725	48.7705	71.6296	83.3846	94.1608
	$10^{2}$	21.3505	43.7697	49.4744	72.1108	83.7983	94.5273
	$10^{3}$	32.8367	50.3805	55.4087	76.3045	87.4332	97.7641
SSSS	0	16.4152	41.0379	41.0379	65.6607	82.0759	82.0759
	$10^{2}$	18.4015	41.8721	41.8721	66.1852	82.4961	82.4961
	$10^{3}$	31.0003	48.7409	48.7409	70.7311	86.1859	86.1859
SCSF	0	10.5508	27.4970	34.6794	52.4033	60.2060	75.3527
	$10^{2}$	13.4342	28.7270	35.6626	53.0590	60.7776	75.8102
	$10^{3}$	28.3352	38.0480	43.5227	58.6316	65.6988	79.8097
SSSF	0	9.7169	23.0822	34.2592	49.1190	51.4434	75.0888
	$10^{2}$	12.7896	24.5346	35.2541	49.8180	52.1113	75.5479
	$10^{3}$	28.0354	34.9908	43.1886	55.7157	57.7754	79.5606
SFSF	0	8.0095	13.4177	30.5411	32.3867	38.8676	58.8277
	$10^2$	11.5459	15.7858	31.6531	33.4374	39.7473	59.4125
	$10^{3}$	27.4903	29.5228	40.3029	41.7189	46.9281	64.4380

**Table 4.6:** Non-dimensional fundamental frequencies parameters ( $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$ ) for square isotropic S-FGM plate with levy type boundary conditions for different value of Pasternak parameters for  $K_W = 0$  and h/a = 0.01.

		Mode no.	Mode no.						
BCs	$K_p$	1	2	3	4	5	6		
SCSC	0	24.0756	45.5245	57.6525	78.6574	85.0033	107.3562		
	$10^{2}$	45.4730	74.7470	84.3674	109.4778	118.8832	137.8422		
	$10^{3}$	122.0313	191.8518	199.5380	251.0251	275.7283	290.5662		
SCSS	0	19.6643	42.9725	48.7705	71.6296	83.3846	94.1608		
	$10^{2}$	42.6788	72.8972	77.4132	103.7698	117.5635	126.6066		
	$10^{3}$	119.9277	190.4888	194.2144	246.7194	274.7460	281.9498		
SSSS	0	16.4152	41.0379	41.0379	65.6607	82.0759	82.0759		
	$10^{2}$	40.4296	71.3922	71.3922	98.8519	116.4556	116.4556		

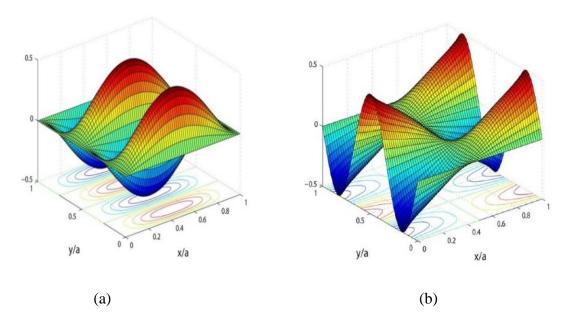
	$10^{3}$	117.9846	189.2389	189.2389	242.7240	273.8448	273.8448
SCSF	0	10.5508	27.4970	34.6794	52.4033	60.2060	75.3527
	$10^{2}$	42.8405	49.4468	73.6196	78.7050	106.7708	119.4213
	$10^{3}$	119.9291	190.4963	194.2309	246.7644	274.7689	282.0378
SSSF	0	9.7169	23.0822	34.2592	49.1190	51.4434	75.0888
	$10^{2}$	40.5463	49.4468	71.9704	72.3635	101.3460	118.0210
	$10^{3}$	117.9859	189.2458	189.2531	242.7641	273.8657	273.9204
SFSF	0	8.0095	13.4177	30.5411	32.3867	38.8676	58.8277
	$10^{2}$	40.6659	49.4934	72.6075	73.4086	104.1469	119.9882
	$10^{3}$	117.9872	189.2526	189.2673	242.8042	273.8866	273.9961

**Table 4.7:** Non-dimensional fundamental frequency parameter  $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$  for square S-FGM plate for different Winkler-Pasternak parameters with different values of sigmoid volume fraction index, k with different levy type boundary conditions and k/a = 0.01.

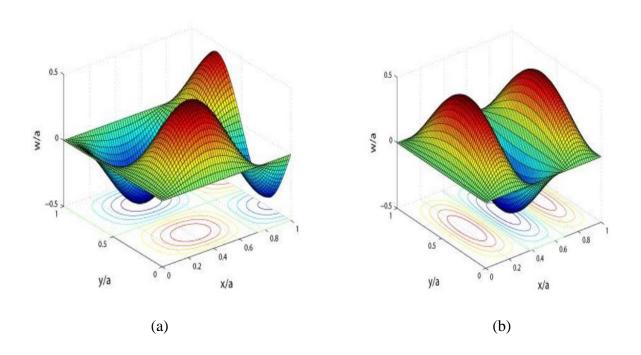
BC's	$K_{\mathbf{w}}$	$K_{\mathbf{P}}$	k = 0	k = 0.1	k = 0.5	k = 1	k = 2	k = 5	k = 10
SCSC	0	0	7.2856	7.2622	6.9976	6.6846	6.3214	5.9953	5.8954
	$10^{2}$	0	7.7080	7.6832	7.4032	7.0721	6.6879	6.3429	6.2372
	0	$10^{2}$	13.7608	13.7166	13.2167	12.6255	11.9396	11.3237	11.1350
	$10^2$	$10^2$	13.9890	13.8419	13.4359	12.8349	12.1376	11.5115	11.3197
SCSS	0	0	5.9507	5.9316	5.7154	5.4598	5.1632	4.8968	4.8152
	$10^{2}$	0	6.4609	6.4402	6.2055	5.9279	5.6059	5.3167	5.2281
	0	$10^{2}$	12.9152	12.8737	12.4046	11.8497	11.2060	10.6279	10.4508
	$10^{2}$	$10^{2}$	13.1581	13.1159	12.6379	12.0726	11.4167	10.8278	10.6473
SSSS	0	0	4.9675	4.9515	4.7711	4.5577	4.3101	4.0877	4.0196
	$10^{2}$	0	5.5685	5.5507	5.3484	5.1091	4.8316	4.5824	4.5060
	0	$10^{2}$	12.2345	12.1953	11.7508	11.2252	10.6154	10.0678	9.9000
	$10^{2}$	$10^{2}$	12.4907	12.4506	11.9968	11.4602	10.8376	10.2786	10.1072
SCSF	0	0	3.1928	3.1826	3.0666	2.9294	2.7703	2.6274	2.5836
	$10^{2}$	0	4.0654	4.0523	3.9046	3.7300	3.5273	3.3454	3.2896
	0	$10^{2}$	12.9641	12.9225	12.4516	11.8946	11.2484	10.6682	10.4904
	$10^2$	$10^2$	13.2061	13.1637	12.6840	12.1167	11.4584	10.8673	10.6862

SSSF	0	0	2.9405	2.9310	2.8242	2.6979	2.5513	2.4197	2.3794
	$10^{2}$	0	3.8703	3.8579	3.7173	3.5510	3.3581	3.1849	3.1318
	0	$10^2$	12.2699	12.2305	11.7847	11.2576	10.6460	10.0969	9.9286
	$10^2$	$10^2$	12.5253	12.4851	12.0301	11.4920	10.8677	10.3071	10.1353
SFSF	0	0	2.4238	2.4160	2.3279	2.2238	2.1030	1.9945	1.9613
	$10^2$	0	3.4940	3.4827	3.3558	3.2057	3.0316	2.8752	2.8272
	0	$10^{2}$	12.3061	12.2666	11.8195	11.2908	10.6774	10.1267	9.9579
	$10^{2}$	$10^{2}$	12.5607	12.5204	12.0641	11.5245	10.8984	10.3362	10.1639

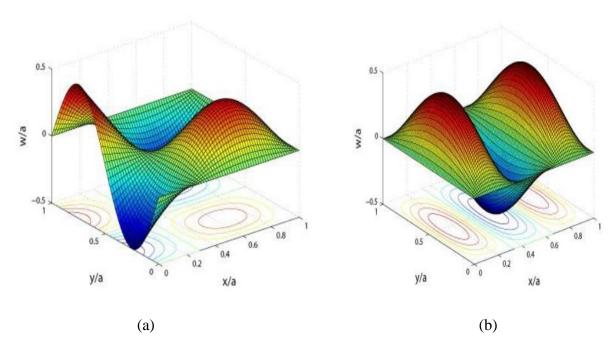
Mode shapes of randomly selected modes of the square FGM plates for all six Levy type boundary conditions are shown in Figs. 4.4, 4.5 and 4.6 where each figure consists of mode shapes of two Levy type boundary conditions. Mode shape for the 9<sup>th</sup> mode (m=4, n=1) of the SCSC edge conditions is shown in Fig. 4.4 (a) whereas Fig. 4.4 (b) shows the same 9<sup>th</sup> mode (m=3, n=2) for the SFSF plate. In Fig. 4.5 (a), mode shape of 4<sup>th</sup> mode (m=2, n=2) with SFSC edge conditions is shown whereas Fig. 4.5 (b) presents 6<sup>th</sup> mode (m=1, n=3) of the SCSS plate. Mode shapes for remaining two Levy type of boundary conditions are shown in Fig. 4.6 for two arbitrary modes. Figs. 4.6 (a) and (b) shows the 4<sup>th</sup> mode and 6<sup>th</sup> mode of the square SFSS and SSSS FGM plates, respectively. Note that, for a particular edge condition, these mode shapes do not change due to the different material properties variations (e.g. S-FGM or P-FGM plates) including the special case of the homogeneous isotropic plate. Only the non-dimensional frequency of these modes will be different when different variations of the material properties are considered.



**Fig. 4.4:** Mode shapes of square FGM plate with two edge conditions: (a) SCSC; 9th mode (m=4, n=1) and (b) SFSF; 9th mode (m=3, n=2).



**Fig. 4.5:** Mode shapes of square FGM plate with two edge conditions: (a) SFSC; 4th mode (m=2, n=2) and (b) SCSS; 6th mode (m=1, n=3).



**Fig. 4.6:** Mode shapes of square FGM plate with two edge conditions: (a) SFSS; 4th mode (m=2, n=2) and (b) SSSS; 5th mode (m=1, n=3)

### 4.4.2.2. Natural frequency of rectangular S-FGM plate with Winkler-Pasternak elastic foundation

A new set of natural frequencies obtained by DSM for a rectangular S-FGM plate is reported in this subsection. The fundamental natural frequencies of rectangular plate for three different aspect ratios for six different sigmoid volume fraction indexes with a different combination of elastic moduli ( $K_w$ ,  $K_P$ ) for six different Levy type boundary conditions are presented in Tables (4.8-4.13). It is noticed that the natural frequency decreases with an increase in sigmoid volume fraction index, k for a given aspect ratio and elastic modulus. It is also observed that the natural frequency increases with the increase in the aspect ratio and the elastic moduli of Winkler-Pasternak foundation for a given value of k for all six boundary conditions except for SFSF boundary condition. This increase in natural frequency is because the stiffness of the plate increases due to change in boundary conditions, geometry and elastic modulus of the Winkler-Pasternak foundation. But in the case of free (SFSF) boundary condition, as the aspect ratio increases, the plate becomes less stiff, due to which the natural frequency decreases as in Table 4.13.

**Table 4.8:** Non-dimensional fundamental frequency parameter  $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$  for SCSC rectangular S-FGM plate with different combination of  $(K_w, K_P, k)$  and h/a = 0.01.

a/b	$(K_w, K_P)$	k = 0	k = 0.1	k = 0.5	k = 1	k = 2	k = 5	k = 10
0.5	(0,0)	11.3811	11.3446	10.9311	10.4422	9.8749	9.3655	9.2094
	$(10^2, 10^2)$	32.7366	32.6315	31.4422	30.0359	28.4041	26.9389	26.4899
	$(10^3, 10^3)$	97.1764	96.8645	93.3343	89.1596	84.3158	79.9665	78.6336
1.5	(0,0)	46.8592	46.7088	45.0065	42.9934	40.6577	38.5604	37.9177
	$(10^2, 10^2)$	69.3172	69.0946	66.5765	63.5987	60.1435	57.0411	56.0903
	$(10^3, 10^3)$	165.9049	165.3723	159.3453	152.2181	143.9486	136.5231	134.2475
2	(0,0)	79.2205	78.9662	76.0883	72.6850	68.7363	65.1906	64.1040
	$(10^2, 10^2)$	101.8234	101.4965	97.7975	93.4232	88.3478	83.7905	82.3938
	$(10^3, 10^3)$	212.9899	212.3062	204.5687	195.418	184.8022	175.2694	172.3479

**Table 4.9:** Non-dimensional fundamental frequency parameter  $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$  for SCSS rectangular S-FGM plate with different combinations of  $(K_w, K_P, k)$  and h/a = 0.01.

a/b	$(K_w,K_P)$	k = 0	k = 0.1	k = 0.5	k = 1	k = 2	k = 5	k = 10
0.5	(0,0)	10.7431	10.7086	10.3184	9.8568	9.3213	8.8405	8.6932
	$(10^2, 10^2)$	32.3741	32.2702	31.0941	29.7034	28.0897	26.6407	26.1966
	$(10^3, 10^3)$	96.8740	96.5631	93.0438	88.8822	84.0535	79.7176	78.3889
1.5	(0,0)	35.3662	35.2527	33.9679	32.4486	30.6858	29.1029	28.6178
	$(10^2, 10^2)$	61.0886	60.8925	58.6733	56.0489	53.0040	50.2698	49.4319
	$(10^3, 10^3)$	158.8816	158.3715	152.5997	145.7742	137.8547	130.7437	128.5643
2	(0,0)	57.6525	57.4675	55.3731	52.8963	50.0226	47.4423	46.6515
	$(10^2, 10^2)$	84.7763	84.5041	81.4244	77.7824	73.5567	69.7624	68.5996
	$(10^3, 10^3)$	201.2634	200.6173	193.3058	184.6597	174.6277	165.6197	162.8590

**Table 4.10:** Non-dimensional fundamental frequency parameter  $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$  for SSSS rectangular S-FGM plate with different combinations of  $(K_w, K_P, k)$  and h/a = 0.01.

a/b	$(K_w, K_P)$	k = 0	k = 0.1	k = 0.5	k = 1	k = 2	k = 5	k = 10
0.5	(0,0)	10.2595	10.2266	9.8538	9.4131	8.9017	8.4425	8.3018
	$(10^2, 10^2)$	32.0561	31.9532	30.7887	29.4116	27.814	26.3790	25.9393
	$(10^3, 10^3)$	96.5848	96.2748	92.7660	88.6168	83.8025	79.4797	78.1548
1.5	(0,0)	26.6747	26.5890	25.6200	24.4741	23.1445	21.9506	21.5847
	$(10^2, 10^2)$	54.7628	54.5870	52.5976	50.2450	47.5154	45.0643	44.3132
	$(10^3, 10^3)$	153.6833	153.1899	147.6069	141.0048	133.3444	126.4660	124.3579
2	(0,0)	41.0379	40.9062	39.4154	37.6524	35.6069	33.7701	33.2072
	$(10^2, 10^2)$	71.8749	71.6441	69.0331	65.9454	62.3627	59.1458	58.1600
	$(10^3, 10^3)$	191.0574	190.4440	183.5033	175.2956	165.7723	157.2211	154.6005

**Table 4.11:** Non-dimensional fundamental frequency parameter  $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$  for SFSC rectangular S-FGM plate with different combinations  $(K_w, K_P, k) h/a = 0.01$ .

a/b	$(K_w,K_P)$	k = 0	k = 0.1	k = 0.5	k = 1	k = 2	k = 5	k = 10
0.5	(0,0)	8.6699	8.5787	8.3271	7.9546	7.5225	7.1344	7.0155

	$(10^2, 10^2)$	32.3858	32.2818	31.1053	29.7140	28.0998	26.6503	26.2060
	$(10^3, 10^3)$	96.8741	96.5632	93.0439	88.8823	84.0536	79.7177	78.3890
1.5	(0,0)	13.9896	13.8426	13.4366	12.8356	12.1382	11.5121	11.3202
	$(10^2, 10^2)$	62.0314	61.8323	59.5788	56.9140	53.8220	51.0457	50.1948
	$(10^3, 10^3)$	158.8905	158.3804	152.6083	145.7824	137.8625	130.7510	128.5715
2	(0,0)	18.9737	18.7742	18.2235	17.4084	16.4627	15.6135	15.3532
	$(10^2, 10^2)$	88.3102	88.0267	84.8186	81.0248	76.6230	72.6705	71.4592
	$(10^3, 10^3)$	201.3016	200.6554	193.3425	184.6947	174.6608	165.6511	162.8899

**Table 4.12:** Non-dimensional fundamental frequency parameter  $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$  for SSSF rectangular S-FGM plate with different combinations of  $(K_w, K_P, k)$  and h/a = 0.01.

a/b	$(K_w, K_P)$	k = 0	k = 0.1	k = 0.5	k = 1	k = 2	k = 5	k = 10
0.5	(0,0)	8.5648	8.4748	8.2262	7.8582	7.4313	7.0480	6.9305
	$(10^2, 10^2)$	32.0662	31.9632	30.7983	29.4208	27.8224	26.3872	25.9474
	$(10^3, 10^3)$	96.5849	96.2749	92.7661	88.6169	83.8026	79.4797	78.1549
1.5	(0,0)	11.4022	11.2823	10.9514	10.4615	9.8939	9.3829	9.2265
	$(10^2, 10^2)$	50.1446	49.9836	48.1620	46.0078	43.5083	41.2640	40.5762
	$(10^3, 10^3)$	153.6904	153.1971	147.6138	141.0113	133.3506	126.4719	124.3638
2	(0,0)	13.4178	13.2767	12.8873	12.3108	11.6420	11.0415	10.8574
	$(10^2, 10^2)$	50.1872	50.0261	48.2029	46.0469	43.5453	41.2990	40.6106
	$(10^3, 10^3)$	191.0855	190.4721	183.5303	175.3214	165.7967	157.2443	154.6232

**Table 4.13:** Non-dimensional fundamental frequency parameter  $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$  for SFSF rectangular S-FGM plate with different combinations  $(K_w, K_P, k) h/a = 0.01$ .

a/b	$(K_w, K_P)$	k = 0	k = 0.1	k = 0.5	k = 1	k = 2	<i>k</i> = 5	k = 10
0.5	(0,0)	8.0967	8.0116	7.7766	7.4287	7.0251	6.6628	6.5517
	$(10^2, 10^2)$	32.0762	31.9733	30.8080	29.4300	27.8312	26.3955	25.9556
	$(10^3, 10^3)$	96.5850	96.2750	92.7662	88.6170	83.8027	79.4798	78.1550

1.5	(0,0)	8.8486	8.8250	7.9343	7.6929	7.3967	6.7409	6.5919
	$(10^2, 10^2)$	49.5395	49.3805	47.5808	45.4526	42.9833	40.7661	40.0865
	$(10^3, 10^3)$	153.6976	153.2042	147.6207	141.0179	133.3569	126.4778	124.3696
2	(0,0)	9.9806	9.8274	8.5978	8.2580	7.8637	7.5096	7.4011
	$(10^2, 10^2)$	50.0556	49.9013	48.1555	46.0911	44.6958	42.5449	40.8858
	$(10^3, 10^3)$	191.1136	190.5001	183.5574	175.3472	165.8211	157.2674	154.6460

#### 4.4.3. A parametric investigation

In this section, the effect of different parameters such as modulus ratio,  $(E_{rat})$ , density ratio ( $\rho_{rat}$ ), and Winkler and Pasternak modulus  $(K_{w}, K_{P})$  and aspect ratio (a/b) on natural frequency are reported.

The first six non-dimensional frequency parameters of a square S-FGM plate for SCSC boundary conditions with different  $E_{rat}$ , for different combinations of Winkler and Pasternak modulus  $(K_{w}, K_{P})$ , for a given sigmoid volume fraction index, k = 2 and given density ratio  $\rho_{rat} = 2$  are shown in Fig. 4.7 (a-d). It is observed that natural frequencies of a given mode decrease as  $E_{rat}$  increase for a given combination of Winkler and Pasternak modulus  $E_{rat}$  increases, stiffness of the plate decreases due to more metal constituents is introduced in the S-FGM plate. A similar trends in variation of natural frequency are observed by authors [259, 260].

It also noted that for a given value of modulus ratio, the natural frequency increases when Winkler and Pasternak modulus  $(K_w, K_P)$  is introduced. The influence of Pasternak modulus on natural frequency variation is more than the Winkler modulus as shown in Fig. 4.7 (b-d), a similar observation is also reported by authors [259, 260].

The influence of density ratio,  $\rho_{rat}$  on non-dimensional frequency parameters of a square S-FGM plate for SCSC boundary condition with different combination of Winkler and Pasternak modulus  $(K_w, K_P)$ , for given value of sigmoid volume fraction index k=2 and for a given value of modulus ratio  $E_{rat}=2$ , are shown in Fig. 4.8 (a-b). It is noticed that natural frequency parameters are affected by the density ratio and the Winkler and Pasternak modulus  $(K_w, K_P)$  values. The increase in the natural frequency parameters are significant for smaller density ratio range, -(say up to  $\rho_{rat}=10$ ) for a given Winkler and Pasternak

modulus. For high density ratio ( $\rho_{rat} > 10$ ), increases the natural frequency parameters is not significant as shown in Fig. 4.8 (a-b). In the Fig. 4.8 (c), shows that the influence of same values of  $E_{rat}$  and  $\rho_{rat}$  ( $E_{rat} = \rho_{rat}$ ) on natural frequency for square S-FGM plate with constant values of k = 2, and ( $K_{w}, K_{P} = 100,100$ ) under SCSC boundary condition. It is obtained from the figure that natural frequency decreases significant at low value of  $E_{rat} = \rho_{rat} = 10$  and for high value of  $E_{rat} = \rho_{rat} > 10$ , decreases the natural frequency is not significant is shown in Fig. 4.8 (c).

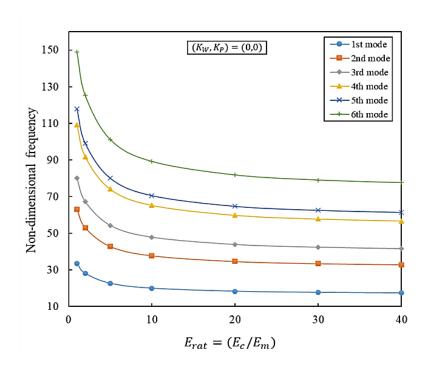
Further, it is apparent that the Pasternak modulus has significant influence on natural frequencies for a given density ratio. These observations are similar to as reported by authors [259, 260].

Fig. 4.9 (a-b) shows that the effect of sigmoid volume fraction index, k for different density ration on natural frequency for a square S-FGM plate for SSSS boundary conditions for a different combinations of Winkler and Pasternak modulus, ( $K_W = 100, K_p = 0$ ) and ( $K_W = 0, K_p = 100$ ) and a given elastic modulus,  $E_{rat} = 2$ .

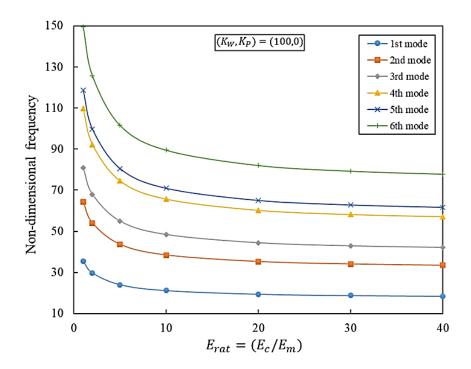
As the trends in variations of natural frequencies for different Levy type boundary conditions (SSSS, SCSC......) are similar, so in the Fig, 4.9 (a-b), a different boundary condition, SSSS is chosen.

It is noticed in Fig. 4.9 (a-b) that the fundamental frequency decreases with increases in sigmoid volume fraction index (k) for a given density ratio and a given Winkler and Pasternak modulus. This is because of reduced flexural stiffness and more metallic constituents' presence in the plate. Further, for very low-density ratio -(say,  $\rho_{rat}$  less than 2), the sigmoid volume fraction index, k does not seem to have significant impact on fundamental frequency.

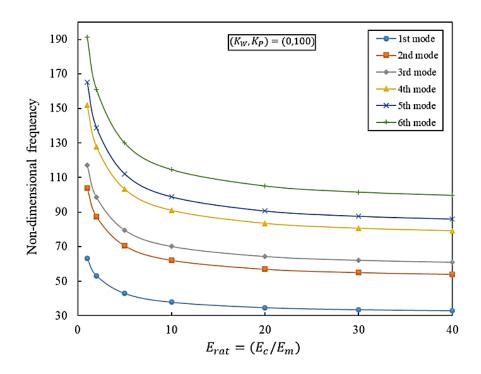
The influence of aspect ratio (a/b) on non-dimensional natural frequency parameters  $(\omega^*)$  with different Winkler-Pasternak modulus for SSSS boundary condition for sigmoid volume fraction index k=1,  $E_{rat}=2$ ,  $\rho_{rat}=2$  and is shown in Fig (4.10-14.12), respectively. It is noticed that as aspect ratio increases, natural frequency increases due to increase the flexural stiffness of the plate. It is observed from these figures that the influence of Pasternak modulus on natural frequency is high then the Winkler modulus.



**Fig. 4.7** (a): Variation of non-dimensional natural frequency parameter ( $\omega^*$ ) with  $E_{rat}$ , for square S-FGM plate for constant  $(K_W, K_p = 0.0)$ , k = 2,  $\rho_{rat} = 2$ , and h/a = 0.01.



**Fig. 4.7 (b):** Variation of non-dimensional natural frequency parameter ( $\omega^*$ ) with  $E_{rat}$ , for square S-FGM plate for constant  $(K_W, K_p = 100,0)$ , k = 2,  $\rho_{rat} = 2$ , and h/a = 0.01.



**Fig. 4.7 (c):** Variation of non-dimensional natural frequency parameter ( $\omega^*$ ) with  $E_{rat}$ , for square S-FGM plate for constant ( $K_W$ ,  $K_p = 0.100$ ), k = 2,  $\rho_{rat} = 2$ , and h/a = 0.01.

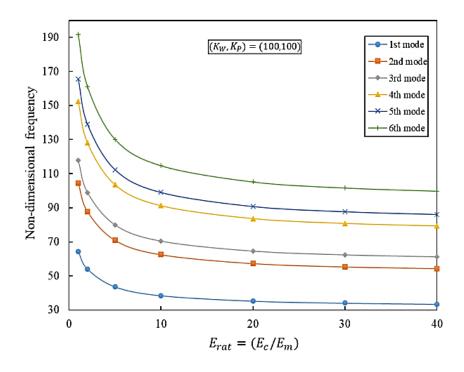
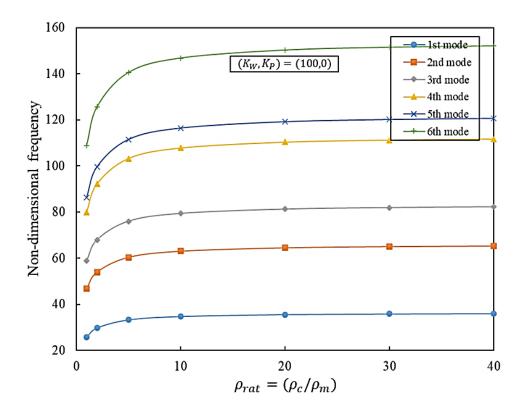
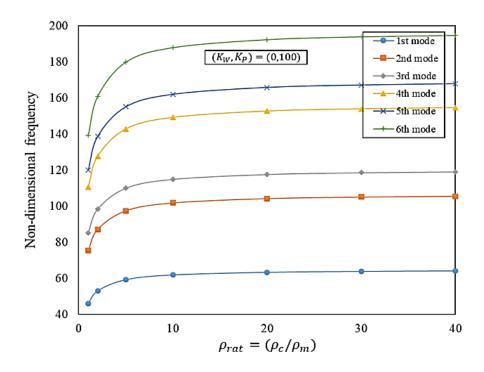


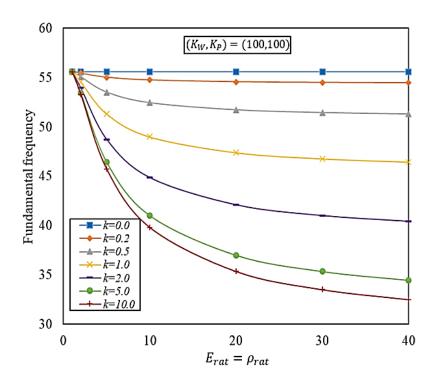
Fig. 4.7 (d): Variation of non-dimensional natural frequency parameter ( $\omega^*$ ) with  $E_{rat}$ , for square S-FGM plate for constant  $(K_W, K_p = 100,100)$ , k = 2,  $\rho_{rat} = 2$ , and h/a = 0.01.



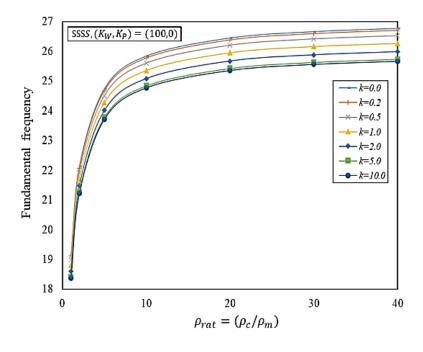
**Fig. 4.8 (a):** Variation of non-dimensional natural frequency parameter ( $\omega^*$ ) with  $\rho_{rat}$ , for square S-FGM plate for constant ( $K_W$ ,  $K_p = 100,0$ ), k = 2,  $E_{rat} = 2$ , and h/a = 0.01.



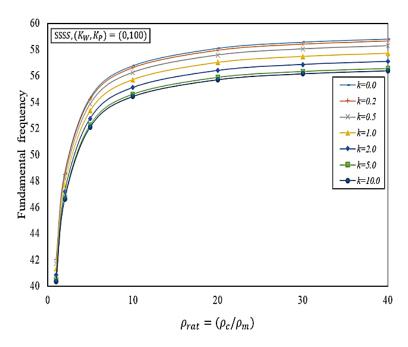
**Fig. 4.8 (b):** Variation of non-dimensional natural frequency parameter ( $\omega^*$ ) with  $\rho_{rat}$ , for square S-FGM plate for constant ( $K_W$ ,  $K_p = 0.100$ ) k = 2,  $E_{rat} = 2$  and h/a = 0.01.



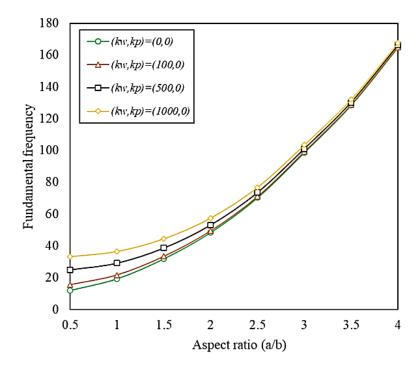
**Fig. 4.8 (c):** Variation of non-dimensional natural frequency parameter ( $\omega^*$ ) with different k and values of square S-FGM plate for constant  $(K_W, K_p = 100, 100)$ ,  $(\rho_{rat} = E_{rat})$  and h/a = 0.01.



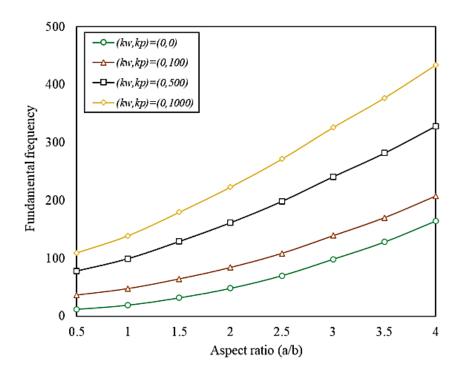
**Fig. 4.9 (a):** Variation of non-dimensional natural frequency parameter ( $\omega^*$ ) with k,  $\rho_{rat}$ , for square S-FGM plate for SSSS boundary condition for constant  $(K_W, K_p = 100,0)$ ,  $E_{rat} = 2$  and h/a = 0.01.



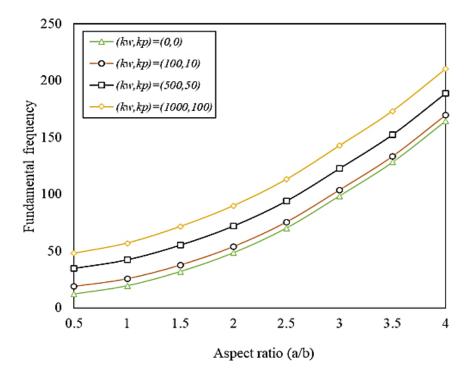
**Fig. 4.9 (b):** Variation of non-dimensional natural frequency parameter ( $\omega^*$ ) with k,  $\rho_{rat}$ , for square S-FGM plate for SSSS boundary condition for constant ( $K_W$ ,  $K_p = 0.100$ ),  $E_{rat} = 2$ , ( $K_W$ ,  $K_p = 0.100$ ) and h/a = 0.01.



**Fig.4.10:** Variation of non-dimensional natural frequency parameter ( $\omega^*$ ) with different aspect ratio for different combinations of  $K_W$  and with fixed  $K_p = 0$  for SSSS boundary conditions for k = 1,  $E_{rat} = 2$ ,  $\rho_{rat} = 2$  and h/a = 0.01.



**Fig. 4.11:** Variation of non-dimensional natural frequency parameter ( $\omega^*$ ) with different aspect ratio for different combinations of  $K_p$  and with fixed  $K_W = 0$  for SSSS boundary conditions for k = 1,  $E_{rat} = 2$ ,  $\rho_{rat} = 2$  and h/a = 0.01.



**Fig. 4.12:** Variation of non-dimensional natural frequency parameter  $(\omega^*)$  with aspect ratio for four different combinations of  $K_W$  and  $K_p$  for SSSS boundary conditions for k=1,  $E_{rat}=2$ ,  $\rho_{rat}=2$  and h/a=0.01.

#### 4.4. Summary

In this chapter of the thesis, the dynamic stiffness method has been developed for thin S-FGM plate embedded on Winkler and Pasternak foundation. This work demonstrates that the DSM combined with the Wittrick and Williams algorithm can be used effectively to calculate the fundamental natural frequency of the S-FGM plate embedded on the Winkler and Pasternak foundation. The physical neutral surface instead of a geometrical mid surface is implemented to model the S-FGM plate. These natural frequencies obtained by DSM for thin S-FGM plate embedded on Winkler and Pasternak foundation are validated with published literature and observed that they are in excellent agreement. This present work contributes a new set of natural frequency results incorporating the elastic foundation for square and rectangular S-FGM plates. The effect of different material parameters, levy type boundary conditions, and elastic foundation modulus on the natural frequencies is also highlighted in different tables and graphs.

The natural frequency of S-FGM plate embedded on elastic foundation is more than the without foundation of the plate. Here, the Pasternak modulus with different material property parameters (sigmoid volume fraction index, aspect ratio, boundary condition) has more influence on increasing the natural frequency than the Winkler modulus.

Observing that obtained natural frequencies are satisfactorily accurate for S-FGM plate embedded on elastic foundation and these results can also be used as a benchmark solution for validation purposes in the future.

### **CHAPTER 5**

# Free Vibration Analysis of E-FGM Plate Resting on Elastic Foundation

#### 5.1 Introduction

In this chapter, it can be concluded from the literature in Chapter 2 that the formulation of the dynamic stiffness method for free vibration analysis of thin E-FGM plates resting on elastic foundation is not well reported. Therefore, continuing from previous work, the dynamic stiffness method (DSM) has been used to investigate the natural vibration of an exponential functionally graded plate (E-FGM) within the framework of Kirchhoff's plate theory (or Classical plate theory) and resting on Winkler and Pasternak elastic foundation. This study similar to the previous work but the only differences between that the material property variation of the FGM plate. Here, the variation of material property continuously varies along the transverse direction of the E-FGM plate by applying exponential law. The governing partial differential equation of motion is derived by implementing Hamilton's principle based on the physical neutral surface of the FGM plate. The application of Wittrick-Williams is applied as a solution method to solve the complex nature of the dynamic stiffness matrix and compute the natural vibration frequencies of the E-FGM rectangular plates. The study of different numerical parameters values (exponential volume fraction index, aspect ratio, boundary conditions and density ratio, modulus ratio, elastic foundation parameters) on natural frequency are also reported. The obtained natural frequency results are compared and validated with the existing available literature. Finally, this study presents a new set of DSM frequency results based on different boundary conditions for E-FGM plates resting on an elastic foundation.

#### 5.2. Contributions and relevant scope

In the present work, the natural vibration behavior in the thickness direction of the E-FGM plate resting on the Winkler and Pasternak elastic foundation is analyzed, where the material property variation of the FGM plate is explained by using exponential law. Kirchhoff's plate theory is used to describe the E-FGM plate's displacement component or kinematic variables. Hence, the effect of shear deformation of the plate can be neglected, and present work is

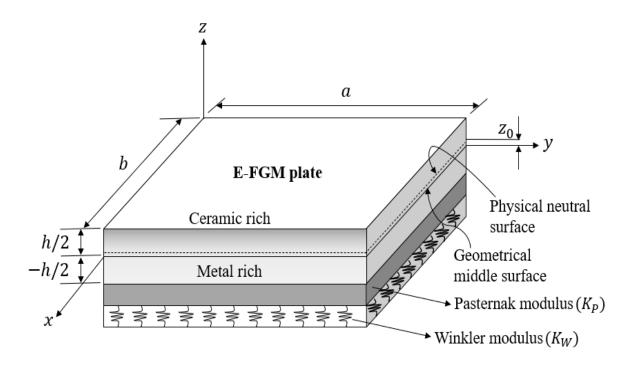
mainly focused on thin E-FGM plates. The Levy-type (displacement and force) boundary condition is implemented where two opposite sides of the plate are simply supported, and the other two sides are arbitrary conditions (free, clamped, and simply supported). Under the present study's relevant scopes and limitations, the thesis main contributions can be explained as follows.

- The development of DSM to analyze the natural vibration behavior of E-FGM plate resting on Winkler and Pasternak elastic foundation is reported.
- A well-known W-W algorithm is implemented to extract the natural frequency of the E-FGM plate.
- For modeling the E-FGM plate, physical neutral surface (PNS) rather than midgeometry of the plate is applied in this present work.
- When comparing DSM frequency results with the available literature, it is noticed that
  obtained results are more accurate and can be used for validation purposes of future
  references.
- This present study also reported some inaccurate published results and tried to explain the possible reasons behind these inaccurate results.
- For different density ratios, elastic modulus ratio, material gradient indices, and aspect
  ratios, a new set of natural frequencies results based on exponential-law FGM plate
  resting on an elastic foundation are highlighted.

#### **5.3.** Theoretical modeling of DSM

#### 5.3.1 Material properties and geometry of E-FGM plate

Fig. 5.1 represents the Cartesian coordinate system of a thin rectangular E-FGM plate where the geometric parameters of the plate are explained as length (a), width (b), and plate thickness (h). The FGM plate is made from the combination of ceramic-metal constituents, where the top surface of the FGM plate is ceramic-rich, and the bottom surface of the plate is metal-rich. The material properties of the FGM plate vary along through the transverse direction by implementing the exponential-law [42, 102] in terms of volume fraction as explained by Eq. (5.1).



**Fig. 5.1:** Schematic representation of thin rectangular exponential-law FGM plate on elastic foundation

As per exponential-law function [42, 102] the variation of material property along the transverse direction is given by

$$P(z) = P_c e^{-\delta \left(1 - \frac{2z}{h}\right)} \quad \text{with } \delta = \frac{1}{2} \ln \left(\frac{P_c}{P_m}\right)$$
 (5.1)

where P(z) represents the material property through the transverse direction of the FGM plate and  $P_c$  and  $P_m$ , indicate the value of material properties of ceramic and metal constituents, respectively. It is noticed from Eq. (5.1) that at  $\delta = 0$ , the value of  $P_c = P_m$  and it is considered as if E-FGM plate behaves like a homogenous isotropic plate with the presence of ceramic constituent. The variation of Young's modulus through the transverse direction of the plate as described in Eq. (5.1) is shown in Fig.1.4 (c) in Chapter 1.

Similar to previous work of section 3.2.1 of Chapter 3, the Poisson's ratio material property variation through the transverse direction is not considered because here FGM plate is taken to be thin (as per Kirchhoff's plate theory), and Poisson's ratio is applied material property variation may become neglected [101, 102]

The detail description of this mathematical model is described in the Chapter 3 and material properties ( $Al_2O_3/Al$ ) are considered as same as that of P-FGM plate. Refer Section 3.2.1 of Chapter 3 for better understanding of Young's modulus variation of the E-FGM plate.

As similar as Section 3.3.2 of Chapter 3, this present study applied Winkler and Pasternak elastic foundation for analyzing the free vibration response of E-FGM plate and the detail explanation of the Winkler-Pasternak model in the FGM plate can be described in Section 3.2.2 of Chapter 3.

#### 5.3.2 Explanation of the physical neutral surface

Here, Kirchhoff's plate theory together with the physical neutral surface (PNS) to examine the displacement components of the given plate same as explained in Chapter 3. The detail explanation of PNS of FGM plate along the thickness direction of the material property is described in the Section 3.2.3 of Chapter 3.

To examine the PNS  $(z_0)$ , the plate total axial force in the particular x, y direction should be zero. Therefore

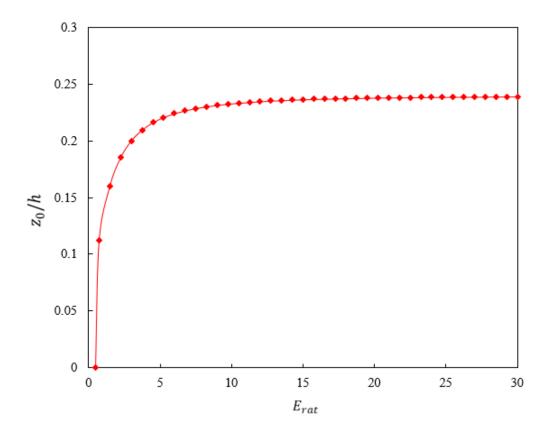
$$\sum F_x = \int_{-h/2 - z_0}^{h/2 - z_0} \sigma_{xx} dA = 0 \tag{5.2}$$

which gives to

$$(z_0) = \frac{\int_{-h/2}^{h/2} E(z)zdz}{\int_{-h/2}^{h/2} E(z)dz}$$

The mathematic expression of the PNS  $(z_0)$  of the E-FGM plate can be expressed as

$$(z_0) = h \left[ \frac{-1}{\log(\frac{E_c}{E_m})} + \frac{E_c + E_m}{E_c - E_m} \right] = h \left[ \frac{-1}{\log(E_{rat})} + \frac{E_{rat} + 1}{E_{rat} - 1} \right]$$
(5.3)



**Fig. 5.2:** Shifting of nondimensional  $(z_0/h)$  parameter with Young's modulus ratio  $(E_{rat})$ .

The mathematical nondimensional expression of PNS of E-FGM plate is represented in Eq. (5.3), which depends upon Young's modulus ratio  $E_{rat} = \frac{E_c}{E_m}$ . The variation of  $E_{rat}$  on a neutral surface is shown in Fig. 5.2. For a particularly given value of Young's modulus ratio  $(E_{rat} = 1)$  with  $z_0 = 0$ , the exponential-law FGM plate is converted to a homogenous isotropic plate, and at this condition, PNS exactly (coincides) to the mid surface of the plate. Eq. (5.3) shows that, as Young's modulus ratio  $(E_{rat})$  increases, the corresponding  $z_0$  value also increases. It is obtained from Fig.5.2 that as  $E_{rat}$  increases, the PNS is moved away from the middle surface of the FGM plate and shifted as for the highly ceramic side of the plate. This phenomenon is due to the higher stiffness value of ceramic constituents at the top part than metal constituents at the bottom part of the exponential-law FGM plate.

#### 5.3.3 The governing free vibration equation of motion of E-FGM plate

The formulation of free vibration equation of motion of E-FGM plate follow the same procedure of P-FGM plate as explained in Section 3.2.4 of Chapter 3, and can be used to obtain the governing differential equation (GDE) for free vibration of the plate. It is noticed that the same natural boundary condition and governing differential equation are obtained and

described in Eq. 3.15 and 3.16 of the previous Chapter 3. For continuation of present thesis work, these equations can be rewritten as

$$D_{FGM}\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + I_0\frac{\partial^2 w}{\partial t^2} + k_W\frac{\partial^2 w}{\partial t^2} + k_P\left(\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\partial^4 w}{\partial y^2 \partial t^2}\right) = 0$$
 (5.4)

The natural boundary conditions on plate element are given as shown in Fig. (3.3) of section 3.2.4 of Chapter 3.

$$V_{x} = \left[ -D_{FGM} \left( \frac{\partial^{3} w}{\partial x^{3}} + (2 - v) \frac{\partial^{3} w}{\partial x \partial y^{2}} \right) \right] \delta w$$

$$M_{xx} = -D_{FGM} \left( \frac{\partial^{2} w}{\partial x^{2}} + v \frac{\partial^{2} w}{\partial y^{2}} \right) \delta \phi_{y}$$
(5.5)

where  $D_{FGM}$  represents the E-FGM plates flexure rigidity,  $V_x$ , and  $M_{xx}$  represents the shear force and bending moment of the FGM plate. The corresponding mathematical expressions can be expressed in Eq. 5.6 and 5.7.

$$(D_{FGM})_{E} = \int_{-h/2-z_{0}}^{h/2-z_{0}} (z_{ns}^{2}) Q_{11}(z_{ns}) dz_{ns} = \int_{-h/2}^{h/2} Q_{11}(z) (z - z_{0})^{2} dz$$

$$= \frac{h^{3}}{\log (E_{c}/E_{m})(1-v^{2})} \left[ \frac{E_{c}-E_{m}}{4} - \frac{E_{m}+E_{c}}{\log (E_{c}/E_{m})} + \frac{2(E_{c}-E_{m})}{\log (E_{c}/E_{m})^{2}} + 2\left(\frac{z_{0}}{h}\right) \left\{ \frac{E_{m}+E_{c}}{2} - \frac{E_{c}-E_{m}}{\log (E_{c}/E_{m})} \right\} + \left(\frac{z_{0}}{h}\right)^{2} \left\{ E_{c} - E_{m} \right\} \right]$$

$$= \frac{12D_{c}}{\log (E_{rat})E_{rat}} \left[ \frac{E_{rat}-1}{4} - \frac{1-E_{rat}}{\log (E_{rat})} + \frac{2(E_{rat}-1)}{\log (E_{rat})^{2}} + 2\left(\frac{z_{0}}{h}\right) \left\{ \frac{1+E_{rat}}{2} - \frac{1-E_{rat}}{\log (E_{rat})} \right\} + \left(\frac{z_{0}}{h}\right)^{2} \left\{ E_{rat} - 1 \right\} \right]$$

$$(5.6)$$

$$(I_0)_E = \int_{-h/2-z_0}^{h/2-z_0} \rho(z_{ns}) dz_{ns} = \int_{-h/2}^{h/2} \rho(z) dz = h\rho_c \left(\frac{\rho_{rat}-1}{\log(\rho_{rat})}\right)$$
 (5.7)

where  $D_{FGM}$  indicates the flexure rigidity and  $I_o$  represents the interia in the transverse direction of the plate. In the present study, nondimensional parameters  $D_{FGM}/D_C$  are applied where the expression is represented by  $D_C = E_c h^3/12(1-\nu^2)$ . At a particular value of Young's modulus ratio  $E_{rat} = 1$ , the E-FGM is advanced as a highly ceramic-rich plate with  $D_{FGM}/D_C = 1$ , and the corresponding E-FGM plate becomes a homogenous isotropic plate. Fig.5.3 represents the effect of  $D_{FGM}/D_C$  for seven different values of Young's modulus ratio  $(E_{rat})$ . Fig.5.3 shows that the parametric value of  $D_{FGM}/D_C$  decreases as Young's modulus ratio increases, and the FGM plate's metallic constituents increase, which has low value of Young's modulus and flexure stiffness than that of ceramic constituents.

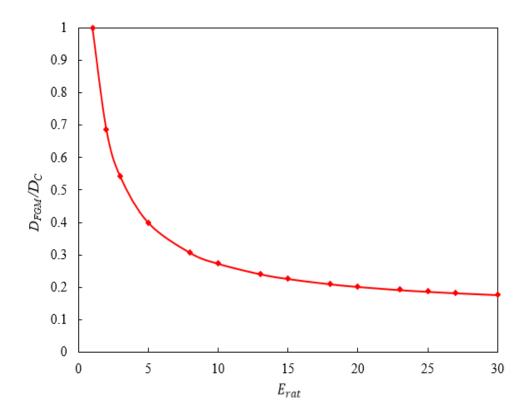


Fig. 5.3 The nondimensional parameter  $D_{EFGM}/D_C$  with different values of  $E_{rat}$ .

Eq. (5.5) represents the natural boundary conditions of the plate element where the shear force  $(V_x)$  of the plate is co-related to the displacement component (w) and associated bending moment is related to the rotation  $\phi_y = \frac{\partial w}{\partial x}$  of the plate. In developing the DS (dynamic stiffness) matrix, the above Eq. (5.4) and Eq. (5.5) are considered main elements and follow the same procedure in section 3.3.3 of Chapter 3 here to obtain the DS matrix for E-FGM plate.

#### 5.3.4 Natural frequency computation using DSM

The development of dynamic stiffness matrix for E-FGM plate resting on elastic foundation follows the same procedures of P-FGM plate as explained in earlier Section .3.2.6 of Chapter 3. A similar type of square 4x4 DS matrix K is formulate with independent six terms which is dependent on free frequency of E-FGM plates. Here, the DS matrixes with six independent terms are dependent on material property variation of exponential law. But in previous these six independent terms are dependent on material property variation of power law, this is the only difference between the both DS matrix in P-FGM and E-FGM plate with the corresponding changes of  $D_{FSM}$  and  $I_o$  terms.

As mentioned earlier that the block assembly formulation of DSM is the same as FEM, but DSM differs in the discretization technique of the structures. In FEM, the assumed shape function is used to discretize the structural element and to obtain separate stiffness and mass matrix, whereas DSM uses eigenvalue-dependent exact shape function to obtain a matrix of a single element that contains both stiffness and mass properties called dynamic stiffness matrix. Consequently, dealing with same penalty method under applied all possible boundary conditions (Levy-type) of the FGM plate. The description of assembly process and penalty method is explained in pervious Section 3.2.7 of Chapter 3.

After completing the assembly process, the obtained global DS matrix is used for estimating the accurate natural frequency results for the E-FGM plates by using the application of Wittrick and Williams algorithm. This algorithm is computationally efficient, accurate, robust and reliable with used the Sturm sequence property of the global matrix and ensure that there is no natural frequency is missed in the given structures. Therefore, the W-W algorithm is well suitable algorithm for DSM and the detail procedure of the algorithm has been explained in the Section 3.2.8 of Chapter 3. In the next sections, the natural frequency DSM results for E-FGM rectangular plates are explained in derails along with proper interpretation and explanations of the obtained results.

#### 5.4. Results and discussions

In this section, the analytical methodology of DSM described above has been imported into the MATLAB software program to extract the E-FGM plate's mode shape and natural frequencies. The boundary conditions of the E-FGM plate, such as clamped, free and simply supported, can be represented as the notation of C, F, and S, respectively and symbols, m, and n denote a particular frequency mode shape, are same as explained in section 3.4 of Chapter 3.

Here, the effective comparative study is presented in this section. First, the natural frequencies formulated by DSM are compared with published results and show very good agreement with them. After that, the effect of design parameters (material gradient index, elastic modulus, material properties ratio, and aspect ratio) at a particular frequency are reported in the form of tables and graphs.

#### .4.1 Comparative study

The following mathematical expression for nondimensional natural frequency parameters are used for comparison of DSM results.

$$\overline{\omega} = \omega h \sqrt{\rho_m/G_m}, \quad \widetilde{\omega} = \omega h \sqrt{\rho_c/E_c}, \qquad \omega^* = \omega a^2 \sqrt{\frac{\rho_c h}{D_c}}, \quad K_p = \frac{k_p a^4}{D_c}, \quad K_w = \frac{k_w a^2}{D_c}$$
 (5.8)

where  $G_m = E_m/(2(1+\nu))$  represents the shear modulus of the metal constituents of the plate. The DSM results of square (a/b = 1.0) and rectangular (a/b = 2.0, 5.0) isotropic plate with simply supported (SSSS) at all the edges for h/a = 0.01 are compared with published literature [273] and are shown in Table 5.1. It is noted that for delta equal to zero, E-FGM plate becomes an isotropic plate. The natural frequencies obtained by DSM are in very good agreement with published results.

**Table 5.1:** Comparison of natural frequencies parameters  $(\overline{\omega} = \omega h \sqrt{\rho_m/G_m})$  of SSSS homogeneous isotropic plate with a different aspect ratio  $(a/b = 1.0, 2.0 \ and 5.0)$ .

a/b=	1.0			a/b=2	2.0			a/b=	5.0		0.01025     0.0102       0.01144     0.0114       0.01341        0.01616		
m								m					
n	Present	Ref.[273]	Ref[273]	m n	Present	Ref.[273]	Ref.[273]	n	Present	Ref.[273]	Ref.[273]		
1 1	0.00079	0.00079	0.00079	11	0.00197	0.00197	0.00197	11	0.01025	0.01025	0.0102		
1 2	0.00197	0.00197	0.00197	2 1	0.00316	0.00316	0.00315	2 1	0.01144	0.01144	0.0114		
2 1	0.00197	0.00197	0.00197	3 1	0.00513	0.00513		3 1	0.01341	0.01341			
2 2	0.00316	0.00316	0.00315	1 2	0.00671	0.00671	0.0067	4 1	0.01616	0.01616			
3 1	0.00395	0.00395	-	2 2	0.00789	0.00789	0.0079	5 1	0.01971	0.01971			
1 3	0.00395	0.00395	0.00394	4 1	0.00789	0.00789	•••	1 2	0.03975	0.03975	0.0393		
3 2	0.00513	0.00513	•••••	3 2	0.00986	0.00986	••	2 2	0.04093	0.04093	0.0405		
23	0.00513	0.00513	0.00512	5 1	0.01144	0.01144		3 2	0.04289	0.04289			
1 4	0.00671	0.00671	•••	4 2	0.01262	0.01262	••	4 2	0.04564	0.04564			
4 1	0.00671	0.00671	•••	1 3	0.01459	0.01459	0.0145	5 2	0.04917	0.04917			
3 3	0.00710	0.00710	0.007109	2 3	0.01577	0.01577	0.0157	13	0.08865	0.08865	0.0866		
2 4	0.00789	0.00789		5 2	0.01616	0.01616		23	0.08982	0.08982	0.0877		
4 2	0.00789	0.00789		3 3	0.01774	0.01774	0.0177	3 3	0.09176	0.09176	0.0895		

The natural frequencies results of the E-FGM plate for two boundary conditions (SSSS, SCSC) obtained by DSM, in Table 3, are compared with those published in the literature for

 $\delta$ =0 and they are in good agreement. It is emphasized that, for  $\delta$ =0, E-FGM behaves as an isotropic plate.

For a special case of E-FGM, for  $\delta = -1/2 \log(P_c/P_m)$ , first seven natural frequencies for SSSS and SCSC edge conditions are compared in Table 3.

**Table 5.2:** Comparison of natural frequencies parameters  $(\overline{\omega} = \omega h \sqrt{\rho_m/G_m})$  of SSSS homogeneous isotropic and E-FGM plate.

SSSS								
Parameter of E-FGM	Source	$\lambda_1$						
$\delta = 0$	Ref.[265]	19.7392	49.3480	49.3480	78.9568	98.6960	98.6960	128.3049
	Ref. [266]	19.7390	49.3480	49.3480	79.4000	100.1700	-	-
	Ref. [267]	19.7390	49.3480	49.3480	78.9570	99.3040	99.3040	-
	Ref. [268]	19.7430	49.3540	49.3540	78.9710	98.7330	-	-
	Ref. [126]	19.7392	49.3490	49.3490	79.4007	100.1729	100.1868	130.3895
	Present	19.7392	49.3480	49.3480	78.9568	98.6960	98.6960	128.3049
$\delta = -1/2\log(P_c/P_m)$	Ref. [126]	15.5462	38.9923	38.9923	62.5293	78.8931	109.9367	132.8760
	Present	13.9055	34.7637	34.7637	55.6219	69.5274	69.5274	90.3856
	%Error	11.7989	12.1638	12.1638	12.4185	13.4705	58.1200	47.0101
SCSC								
$\delta = 0$	Ref.[265]	28.9509	54.7431	69.3270	94.5853	102.2162	129.0955	140.2045
	Ref. [266]	28.9216	54.6658	69.1927	94.3594	101.9944	128.6742	-
	Ref. [267]	28.9460	54.7410	69.3303	94.6120	102.1651	129.0791	-
	Ref. [268]	28.9515	54.7418	69.3261	94.5834	102.2136	129.0915	-
	Ref. [126]	28.9515	54.7418	69.3261	94.5834	102.2136	129.0915	141.3810
	Present	28.9509	54.7431	69.3270	94.5853	102.2162	129.0955	140.2045
$\delta = -1/2\log(P_c/P_m)$	Ref. [126]	22.7996	43.2224	54.6117	74.5803	81.6794	106.3337	128.6259
	Present	20.3947	38.5643	48.8381	66.6315	72.0072	90.9426	98.7685
	%Error	11.7918	12.0788	11.8219	11.9295	13.4323	16.9240	30.2297

It is observed from the Table 5.2 that published frequency results of the E-FGM plate in Ref. [126] are not sufficiently accurate, and the first frequency error is around 11.8%, and for higher mode, the error is around 58.12%. The possible reason for this inaccuracy is that in Ref. [126], the authors considered middle surface instead of physical neutral surface as the reference plane of the E-FGM plate, which is not correct. In our approach, we used physical

neutral surface to compute the natural frequencies. It is further noted that the computation of natural frequencies of E-FGM plate is scanty in the literature.

### 5.4.2 Extraction of the natural frequency of E-FGM plate resting on Winkler-Pasternak foundation

This section highlights the extraction of a new set of natural frequencies for the E-FGM plate developed by DSM under two parametric geometric configurations (one square and another rectangular) resting on the Winkler and Pasternak elastic foundation.

## 5.4.2.1. Extraction of the natural frequency of a square E-FGM plate with Winkler-Pasternak foundation

In this subsection, a new set of natural frequency results of a square E-FGM plate for a different combination of Winkler modulus ( $K_w = 0, 100, 1000$ ) with Levy type edge conditions are shown in Table 5.3. At a particular mode, the natural frequency increases with increases the Winkler modulus value ( $K_w = 0$ , 100, 1000). This is because of stiffness of the E-FGM plate increases. This table shows that the maximum natural frequency value is obtained for SCSC boundary conditions and the minimum for SFSF boundary conditions. The reason is that by adding more constraint at the SCSC edge condition, the stiffness of the plate increases and gives the maximum value of natural frequency while for free edge condition SFSF, removing the constraints at the particular edge which decreases the plate stiffness and gives the minimum value of natural frequency. Similarly, Table 5.4 shows the natural frequency results for Pasternak modulus  $(K_P = 0, 100, 1000)$  for the E-FGM plate under Levy type edge conditions, and natural frequency increases as the Pasternak modulus  $(K_P = 0, 100, 1000)$  of the plate, because of increases the plate stiffness. It is obtained from Tables (5.3-5.5) that the effect of the Pasternak modulus (at  $K_w = 0$ ) is higher than the Winkler modulus(at  $K_P = 0$ ). The obtained mode shapes for the E-FGM plate are the similar as observed in an isotropic plate.

**Table 5.3:** Fundamental frequencies parameters ( $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$ ) for square E-FGM plate under Levy type boundary conditions for different value of Winkler parameters for  $K_p = 0$  and h/a = 0.01.

			E-FGM plate mode no.							
BCs	$K_w$	1	2	3	4	5	6			

SCSC	0	20.3823	38.5408	48.8083	66.5909	71.9633	90.8871
	$10^{2}$	21.5639	39.1785	49.3134	66.9620	72.3068	91.1594
	$10^{3}$	30.1844	44.5090	53.6461	70.2140	75.3284	93.5742
SCSS	0	16.6477	36.3802	41.2888	60.6412	70.5929	79.7160
	$10^{2}$	18.0752	37.0552	41.8847	61.0486	70.9431	80.0263
	$10^{3}$	27.7994	42.6518	46.9087	64.5989	74.0204	82.7665
SSSS	0	13.8970	34.7425	34.7425	55.5880	69.4850	69.4850
	$10^{2}$	15.5786	35.4486	35.4486	56.0320	69.8407	69.8407
	$10^{3}$	26.2447	41.2638	41.2638	59.8806	72.9645	72.9645
SCSF	0	8.9323	23.2788	29.3594	44.3643	50.9701	63.7931
	$10^{2}$	11.3733	24.3201	30.1917	44.9195	51.4540	64.1804
	$10^{3}$	23.9884	32.2112	36.8461	49.6372	55.6202	67.5664
SSSF	0	8.2263	19.5413	29.0037	41.5839	43.5517	63.5697
	$10^{2}$	10.8276	20.7709	29.8459	42.1757	44.1171	63.9584
	$10^{3}$	23.7346	29.6230	36.5633	47.1686	48.9123	67.3556
SFSF	0	6.7808	11.3594	25.8560	27.4184	32.9051	49.8032
	$10^{2}$	9.7747	13.3642	26.7973	28.3079	33.6498	50.2983
	$10^{3}$	23.2731	24.9939	34.1202	35.3189	39.7291	54.5528

**Table 5.4:** Fundamental frequencies parameters ( $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$ ) for square isotropic E-FGM plate under Levy type boundary conditions for different Pasternak parameters for  $K_W = 0$  and h/a = 0.01.

		E-FGM plate mode no.						
BCs	$K_p$	1	2	3	4	5	6	
SCSC	0	20.3823	38.5408	48.8083	66.5909	71.9633	90.8871	
	$10^{2}$	38.4972	63.2804	71.4250	92.6833	100.6459	116.6964	
	$10^{3}$	103.3110	162.4206	168.9277	212.5164	233.4300	245.9917	
SCSS	0	16.6477	36.3802	41.2888	60.6412	70.5929	79.7160	
	$10^{2}$	36.1317	61.7143	65.5376	87.8510	99.5286	107.1845	
	$10^{3}$	101.5301	161.2667	164.4208	208.8712	232.5984	238.6970	
SSSS	0	13.8970	34.7425	34.7425	55.5880	69.4850	69.4850	
	$10^{2}$	34.2274	60.4402	60.4402	83.6874	98.5906	98.5906	

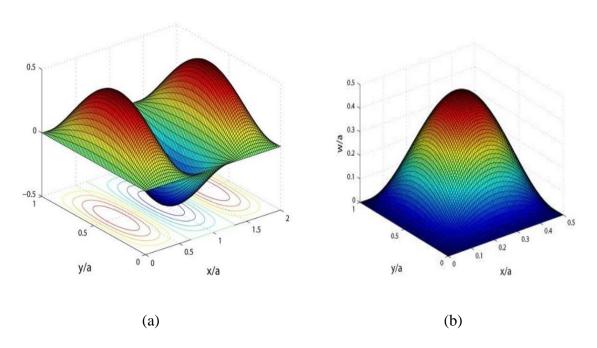
	$10^{3}$	99.8851	160.2086	160.2086	205.4888	231.8354	231.8354
SCSF	0	8.9323	23.2788	29.3594	44.3643	50.9701	63.7931
	$10^{2}$	36.2686	41.8614	62.3260	66.6312	90.3916	101.1014
	$10^{3}$	101.5313	161.2731	164.4347	208.9093	232.6178	238.7716
SSSF	0	8.2263	19.5413	29.0037	41.5839	43.5517	63.5697
	$10^{2}$	34.3263	41.8614	60.9297	61.2625	85.7990	99.9159
	$10^{3}$	99.8862	160.2144	160.2206	205.5227	231.8531	231.8994
SFSF	0	6.7808	11.3594	25.8560	27.4184	32.9051	49.8032
	$10^{2}$	34.4275	41.9008	61.4691	62.1473	88.1701	101.5813
	$10^{3}$	99.8873	160.2201	160.2326	205.5566	231.8708	231.9635

**Table 5.5:** Fundamental frequency parameter  $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$  for square E-FGM plate for different Winkler-Pasternak parameters with different Levy type boundary conditions and h/a = 0.01.

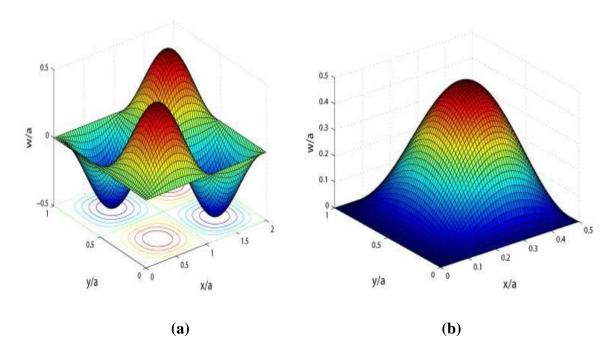
				E-FGM plate mode no.							
BC's	$K_W$	$K_p$	1	2	3	4	5	6			
SCSC	0	0	20.3823	38.5408	48.8083	66.5909	71.9633	90.8871			
	$10^{2}$	0	21.5639	39.1785	49.3134	66.9620	72.3068	91.1594			
	0	$10^{2}$	38.4972	63.2804	71.4250	92.6833	100.6459	116.6964			
	$10^{2}$	$10^{2}$	39.1357	63.6708	71.7711	92.9503	100.8918	116.9086			
SCSS	0	0	16.6477	36.3802	41.2888	60.6412	70.5929	79.7160			
	$10^{2}$	0	18.0752	37.0552	41.8847	61.0486	70.9431	80.0263			
	0	$10^{2}$	36.1317	61.7143	65.5376	87.8510	99.5286	107.1845			
	$10^{2}$	$10^{2}$	36.8112	62.1146	65.9146	88.1326	99.7772	107.4154			
SSSS	0	0	13.8970	34.7425	34.7425	55.5880	69.4850	69.4850			
	$10^{2}$	0	15.5786	35.4486	35.4486	56.0320	69.8407	69.8407			
	0	$10^{2}$	34.2274	60.4402	60.4402	83.6874	98.5906	98.5906			
	$10^{2}$	$10^{2}$	34.9440	60.8488	60.8488	83.9830	98.8417	98.8417			
SCSF	0	0	8.9323	23.2788	29.3594	44.3643	50.9701	63.7931			
	$10^{2}$	0	11.3733	24.3201	30.1917	44.9195	51.4540	64.1804			
	0	$10^{2}$	36.2686	41.8614	62.3260	66.6312	90.3916	101.1014			
	$10^2$	$10^2$	36.9456	42.4493	62.7223	67.0021	90.6653	101.3462			

SSSF	0	0	8.2263	19.5413	29.0037	41.5839	43.5517	63.5697
	$10^{2}$	0	11.3733	24.3201	30.1917	44.9195	51.4540	64.1804
	0	$10^{2}$	34.3263	41.8614	60.9297	61.2625	85.7990	99.9159
	$10^{2}$	$10^{2}$	35.0408	42.4493	61.3351	61.6657	86.0873	100.1636
SFSF	0	0	6.7808	11.3594	25.8560	27.4184	32.9051	49.8032
	$10^{2}$	0	9.7747	13.3642	26.7973	28.3079	33.6498	50.2983
	0	$10^{2}$	34.4275	41.9008	61.4691	62.1473	88.1701	101.5813
	$10^{2}$	$10^2$	35.1400	42.4882	61.8709	62.5448	88.4508	101.8250

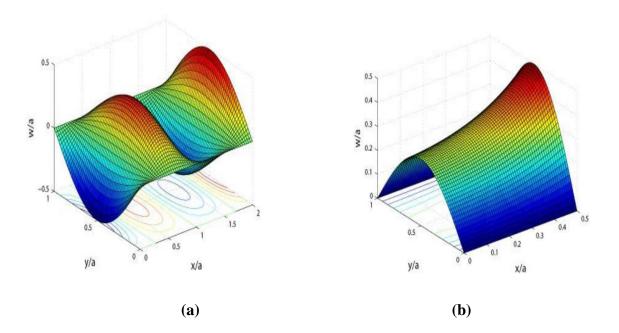
Six figures from Fig. 5.4 to Fig. 5.9 show few randomly chosen mode shapes of rectangular FGM plates with aspect ratios 0.5 and 2.0 for all six Levy type boundary conditions. Mode shapes of  $3^{rd}$  mode (m=1, n=3) and  $1^{st}$  mode (m=1, n=1) of SSSS edge conditions are shown in Fig.5.4 (a) and (b) for aspect ratios 0.5 and 2.0, respectively. Figs. 5.5 (a) and (b) show the mode shapes of 5<sup>th</sup> mode (m=2, n=2) for aspect ratio 0.5 and 1<sup>st</sup> mode (m=1, n=1) for aspect ratio 2.0 with the SCSS edge condition of the FGM plate. Figs. 5.6 (a) and (b) show the 4<sup>th</sup> mode (m=1, n=4) for the aspect ratio 0.5 and 1<sup>st</sup> mode (m=1, n=1) for the aspect ratio 2.0 of the FGM plate with SFSF edge conditions, respectively. Similarly, the other mode shapes shown are:  $4^{th}$  mode (m=2, n=1) for a/b=0.5 and  $1_{st}$  mode (m=1, n=1) for a/b=2.0 for SCSF plate in Figs. 5.7 (a) and (b);  $3^{rd}$  mode (m=1, n=3) for a/b=0.5 and  $5^{th}$  mode (m=2, n=2) for a/b=2.0 for SCSC plate in Figs. 5.8 (a) and (b);  $3^{rd}$  mode (m=1, n=3) for a/b=0.5 and  $4^{th}$  mode (m=4, n=1) for a/b=2.0 for SCSF plate in Figs. 5.9 (a) and (b); It is observed from these figures that significant variation in the mode shapes are possible and the nature of mode shapes greatly depend on the edge conditions and aspect ratios of the FGM plates. However, as discussed earlier, these mode shapes remain unchanged due to different variations of the material properties (i.e. S-FGM or E-FGM plates) including the homogeneous isotropic plate.



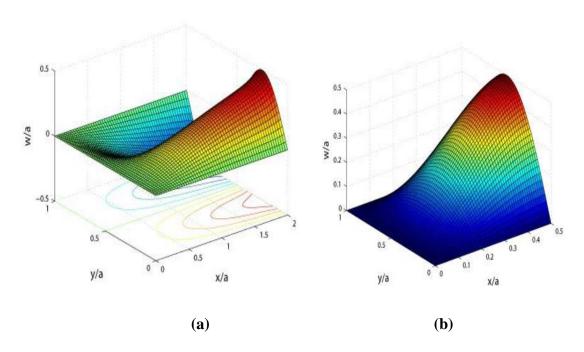
**Fig. 5.4 :** Mode shapes of rectangular FGM plate with SSSS edge condition for aspect ratios: (a) a/b=0.5; 3rd mode (m=1, n=3) and (b) a/b=2.0; 1st mode (m=1, n=1).



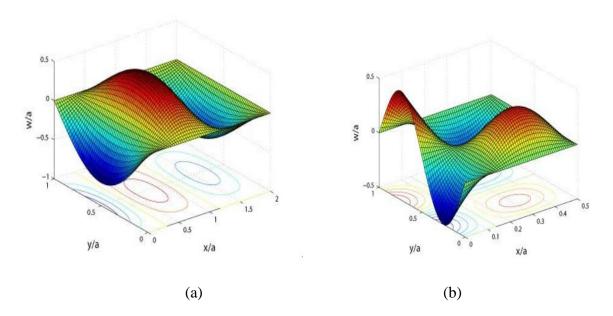
**Fig. 5.5 :** Mode shapes of rectangular FGM plate with SCSS edge condition for aspect ratios: (a) a/b=0.5; 5th mode (m=2, n=2) and (b) a/b=2.0; 1sh mode (m=1, n=1).



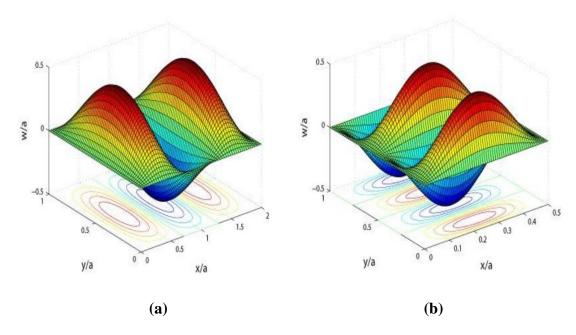
**Fig. 5.6:** Mode shapes of rectangular FGM plate with SFSF edge condition for aspect ratios: (a) a/b=0.5;  $4^{th}$  mode (m=1, n=4) and (b) a/b=2.0; 1st mode (m=1, n=1).



**Fig. 5.7:** Mode shapes of rectangular FGM plate with SCSF edge condition for aspect ratios: (a) a/b=0.5;  $4^{th}$  mode (m=2, n=1) and (b) a/b=2.0;  $1^{st}$  mode (m=1, n=1).



**Fig. 5.8:** Mode shapes of rectangular FGM plate with SFSS edge condition for aspect ratios: (a) a/b=0.5; 3rd mode (m=1, n=3) and (b) a/b=2.0; 5th mode (m=2, n=2).



**Fig. 5.9:** Mode shapes of rectangular FGM plate with SCSC edge condition for aspect ratios: (a) a/b=0.5; 3rd mode (m=1, n=3) and (b) a/b=2.0; 4th mode (m=4, n=1).

## 5.4.2.2. Extraction of the natural frequency of rectangular E-FGM plate resting on Winkler-Pasternak elastic foundation

This subsection reports the fundamental natural frequencies of the E-FGM rectangular plate obtained by DSM. Table (5.6-5.11) represent the fundamental frequency results for three different value of aspect ratios for Levy type edge conditions with different combination of elastic foundation ( $K_w$ ,  $K_P$ ).

It is illustrated from these tables that the frequency of the plate decreases with an increase in the particular value of elastic foundation and aspect ratio. Besides this, as the aspect ratio increases with elastic foundation, the frequency of the plate increases under all boundary conditions except the SFSF condition. As the aspect ratio increases with a change in boundary condition under an elastic foundation, natural frequency increases because the plates stiffness increases. Expect in SFSF boundary condition as aspect ratio increases, the stiffness of the plate decreases because of this natural frequency decreases as seen in Table 5.11.

**Table 5.6:** Fundamental frequency parameter  $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$  for SCSC rectangular E-FGM plate with different combinations of  $(K_w, K_P)$  and h/a = 0.01.

		E-FGM plate mode no.							
a/b	$(K_w,K_P)$	1	2	3	4	5	6		
0.5	(0,0)	9.6352	16.6477	27.2417	29.9823	36.3802	41.2888		
	$(10^2, 10^2)$	27.7146	36.8112	49.7716	55.1187	62.1146	65.9146		
	$(10^3, 10^3)$	82.2691	103.9423	132.8906	149.1406	162.7962	165.9212		
1.5	(0,0)	39.6777	55.6133	86.7224	102.9964	119.7827	149.8483		
	$(10^2, 10^2)$	58.6906	80.0352	115.1113	127.3705	146.2456	178.5592		
	$(10^3, 10^3)$	139.3226	188.4685	253.5702	254.7547	288.3095	339.7268		
2	(0,0)	67.0677	81.5291	110.0800	178.9208	195.2332	223.9523		
	$(10^2, 10^2)$	86.2031	104.9579	137.4472	204.0037	221.6749	252.0986		
	$(10^3, 10^3)$	180.3161	221.8198	280.9727	353.0260	354.3374	382.1766		

**Table 5.7:** Fundamental frequency parameter  $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$  for SCSS rectangular E-FGM plate with different combinations of  $(K_w, K_P)$  and h/a = 0.01.

		E-FGM plate mode no.						
a/b	$(K_w,K_P)$	1	2	3	4	5	6	
0.5	(0,0)	9.09506	15.16031	24.78988	29.73797	35.50507	37.89343	
	$(10^2, 10^2)$	27.40776	35.82716	47.99091	54.93928	61.44974	63.28809	
	$(10^3, 10^3)$	82.01304	103.1233	131.428	148.9958	162.2609	163.7995	
1.5	(0,0)	29.94568	48.5848	81.85974	85.20032	103.9554	136.4582	
	$(10^2, 10^2)$	51.72249	74.73135	111.1315	112.0612	132.478	166.6678	

	$(10^3, 10^3)$	134.4176	184.7955	243.3339	250.7612	278.0799	330.8314
2	(0,0)	48.8083	66.59087	98.70818	146.714	165.1552	196.8829
	$(10^2, 10^2)$	71.7711	92.9503	127.9057	174.7715	176.5969	194.2703
	$(10^3, 10^3)$	170.3885	213.6794	274.4041	332.2075	361.4391	407.8645

**Table 5.8:** Fundamental frequency parameter  $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$  for SSSS rectangular E-FGM plate with different combinations of  $(K_w, K_P)$  and h/a = 0.01.

		E-FGM plate mode no.					
a/b	$(K_w,K_P)$	1	2	3	4	5	6
0.5	(0,0)	8.6856	13.8970	22.5826	29.5311	34.7425	34.7425
	$(10^2, 10^2)$	27.1385	34.9440	46.3619	54.7789	60.8488	60.8488
	$(10^3, 10^3)$	196.2144	232.9020	236.0312	268.7947	271.6440	304.5311
1.5	(0,0)	22.5857	43.4312	69.4975	78.1737	90.3430	125.0855
	$(10^2, 10^2)$	46.3658	70.6649	98.8549	108.0017	120.7139	156.5322
	$(10^3, 10^3)$	130.0262	181.5457	232.9246	248.2810	268.8153	322.8095
2	(0,0)	34.7425	55.5880	90.3305	118.1245	138.9700	173.7124
	$(10^2, 10^2)$	60.8488	83.9830	120.7008	149.3963	170.7223	206.0472
	$(10^3, 10^3)$	161.7481	206.6913	268.7947	312.4295	342.9975	342.9975

**Table 5.9:** Fundamental frequency parameter  $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$  for SFSC rectangular E-FGM plate with different combinations of  $(K_w, K_P)$  and h/a = 0.01.

		E-FGM plate mode no.					
a/b	$(K_w,K_P)$	1	2	3	4	5	6
0.5	(0,0)	7.3399	11.0911	18.1572	28.0384	28.5786	31.7565
	$(10^2, 10^2)$	27.4176	35.8836	48.1786	55.0039	61.7145	63.7505
	$(10^3, 10^3)$	82.0131	103.1239	131.4301	148.9965	162.2639	163.8059
1.5	(0,0)	11.8446	31.8952	42.9654	64.9944	66.0595	99.8273
	$(10^2, 10^2)$	42.4588	52.5211	77.1387	116.4834	118.0765	124.2048
	$(10^3, 10^3)$	134.4251	184.8226	243.4543	250.8352	278.2992	327.5099
2	(0,0)	16.0628	35.7291	69.5421	70.2447	93.1153	128.8514
	$(10^2, 10^2)$	42.5850	74.7629	99.4295	124.6967	140.0169	191.8113

**Table 5.10:** Fundamental frequency parameter  $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$  for SSSF rectangular E-FGM plate with different combinations of  $(K_w, K_P)$  and h/a = 0.01.

		E-FGM plate mode no.							
a/b	$(K_w,K_P)$	1	2	3	4	5	6		
0.5	(0,0)	7.2509	10.3960	16.6298	26.1387	27.9993	31.3462		
	$(10^2, 10^2)$	27.1470	34.9921	42.4492	46.5223	54.8366	61.0862		
	$(10^3, 10^3)$	81.7683	102.3367	130.0192	148.8581	161.7510	161.7541		
1.5	(0,0)	9.6535	30.6768	33.6979	57.3691	65.2589	87.7123		
	$(10^2, 10^2)$	42.4521	46.8585	72.4231	103.3167	112.2665	124.1608		
	$(10^3, 10^3)$	130.0323	181.5686	233.0202	248.3451	268.9973	323.1715		
2	(0,0)	11.3594	32.9051	53.0018	67.6154	78.1652	115.9509		
	$(10^2, 10^2)$	42.4882	62.5448	88.4508	124.4468	130.0714	162.5584		
	$(10^3, 10^3)$	161.7719	206.7588	268.9628	312.8735	343.3472	343.6722		

**Table 5.11:** Fundamental frequency parameter  $\omega^* = \omega a^2 \sqrt{\rho_c h/D_c}$  for SFSF rectangular E-FGM plate with different combinations of  $(K_w, K_P)$  and h/a = 0.01.

		E-FGM plate mode no.							
a/b	$(K_w,K_P)$	1	2	3	4	5	6		
0.5	(0,0)	6.8546	8.2263	12.4508	19.5413	27.5896	29.0037		
	$(10^2, 10^2)$	27.1556	35.0408	42.4493	46.6864	54.8967	61.3351		
	$(10^3, 10^3)$	81.7684	102.3372	130.0212	148.8587	161.7538	161.7600		
1.5	(0,0)	6.7291	15.2327	27.2606	38.6312	46.3923	61.6907		
	$(10^2, 10^2)$	41.9373	42.9864	47.4276	74.5452	108.7395	117.9629		
	$(10^3, 10^3)$	130.1196	181.6532	233.3211	248.4587	269.3672	323.7061		
2	(0,0)	6.6971	19.3764	27.1231	45.4375	61.4512	74.2678		
	$(10^2, 10^2)$	40.6836	44.4664	64.5416	94.1693	120.6637	129.6424		
	$(10^3, 10^3)$	161.7958	206.8265	269.1321	313.3225	343.7025	344.3567		

#### 5.4.3. A parametric study

In this section, the variations of the different values of geometrical parameters such as modulus ratio, density ratio, aspect ratio, and Winkler and Pasternak modulus on the natural frequency of the E-FGM plate are reported.

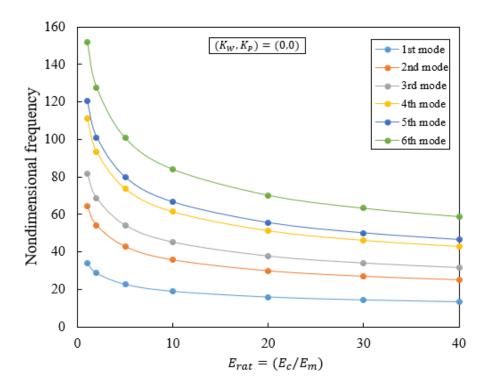
The influence of Winkler and Pasternak elastic modulus ( $K_w$ , $K_P$ ) for different modulus ratios ( $E_{rat}$ ) with a fixed value of density ratio ( $\rho_{rat}=2$ ) on the natural frequency of square E-FGM plate under SCSC edge condition are shown in Fig. 5.10 (a-d). At a particular mode, it can be obtained from these figures that the natural frequency of the E-FGM plate decreases with increases in Young's modulus ratio ( $E_{rat}$ ) because the plate stiffness is decreased and more metal constituents in the E-FGM plate. The influence of Young's modulus ratio of the E-FGM plate natural frequency is similar as obtained by various authors [46,47]. It can be also illustrated that the influence of the Pasternak modulus on natural frequency for a particular mode is significantly high than the Winkler modulus, and similar observation is also reported by authors [47,48]. At a lower value of Young's modulus ratio ( $E_{rat} < 10$ ), the natural frequency decreases for a particular mode with a fixed value of elastic modulus is more evident.

The effect of density ratio ( $\rho_{rat}$ ) for a different combination of elastic modulus ( $K_{w_i}K_P$ ) under a fixed value of Young's modulus ratio ( $E_{rat}=2$ ) on the natural frequency of square E-FGM plate with SCSC edge condition are highlighted in Fig. 5.11 (a-c). As the density ratio increases ( $\rho_{rat}$ ), the frequency of the E-FGM plate increased, and its shows that for the lower value of density ratio ( $\rho_{rat} < 10$ ), increases in natural frequencies are significant for a particular value of Winkler and Pasternak elastic modulus. Clearly, the Pasternak modulus's effect on natural frequency is more significant than the Winkler modulus, and similar observations are reported by authors [259, 260] for other kind of FGM plates.

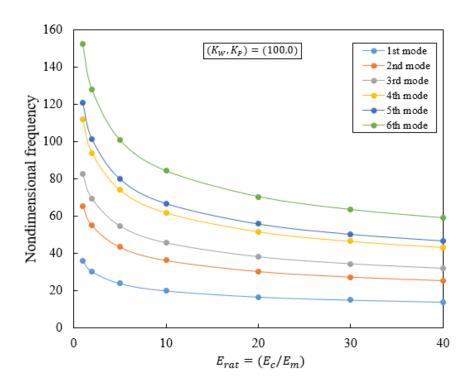
The variation of Winkler-Pasternak elastic modulus  $(K_{w}, K_{P})$  and different values of  $E_{rat} = \rho_{rat}$  on the natural frequency of E-FGM square plate under SCSC edge condition are reported in Fig. 5.12(a-c). It is obtained from these figures as  $E_{rat} = \rho_{rat}$  (0 to 40), the plate natural frequency decreased. At lower value of  $E_{rat} = \rho_{rat}$  (say, less than 10), the decrease in natural frequency is significant.

The effect of Winkler-Pasternak modulus  $(K_{w}, K_{P})$  for different values of aspect ratio (a/b) on the natural frequency of rectangular E-FGM plate with SSSS edge condition and  $E_{rat} = 1$ ,  $\rho_{rat} = 1$  are shown in Fig.5.13 (a-c). A similar pattern of natural frequency variations is obtained for all boundary conditions (Levy-type), therefore, simply supported (SSSS) boundary condition in Levy solution is shown in Fig. 5.13 (a-c).

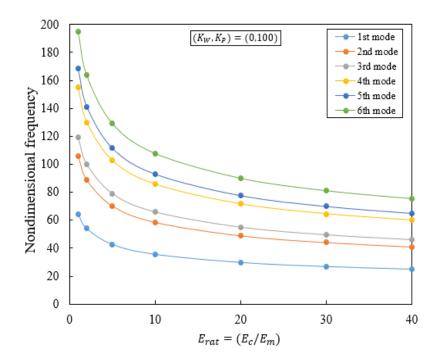
It is noticed from these figures that as the aspect ratio (a/b) of the plate increases, the natural frequency of the E-FGM plate increasing because of increases in the flexural stiffness of the plate. The obtained frequency show that the effect of the Pasternak modulus is higher than the Winkler modulus for all aspect ratio values. It is also illustrated that for the high value of aspect ratio, natural frequency does not change with Winkler modulus, but it increases in the case of Pasternak modulus.



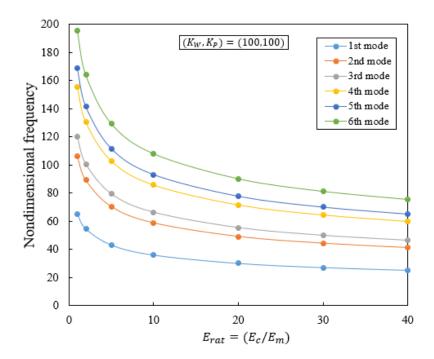
**Fig. 5.10 (a):** Effect of  $E_{rat}$  on frequency parameter ( $\omega^*$ ) for square E-FGM plate for constant  $(K_W, K_p = 0.0)$ ,  $\rho_{rat} = 2$ , and h/a = 0.01.



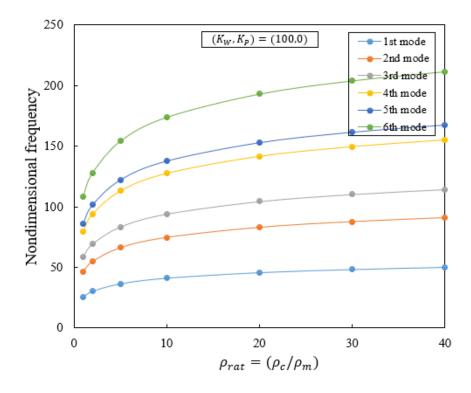
**Fig. 5.10 (b):** Effect of  $E_{rat}$  on frequency parameter ( $\omega^*$ ) for square E-FGM plate for constant  $(K_W, K_p = 100,0)$ ,  $\rho_{rat} = 2$ , and h/a = 0.01.



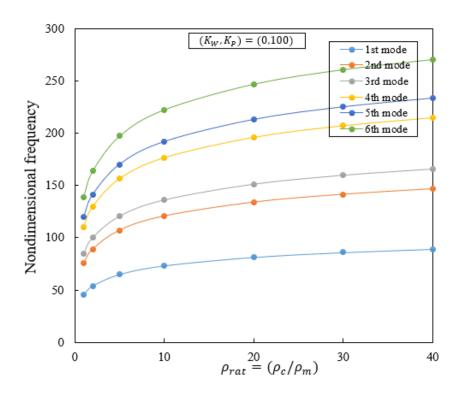
**Fig. 5.10 (c):** Effect of  $E_{rat}$  on frequency parameter ( $\omega^*$ ) for square E-FGM plate for constant  $(K_W, K_p = 0.100)$ ,  $\rho_{rat} = 2$ , and h/a = 0.01.



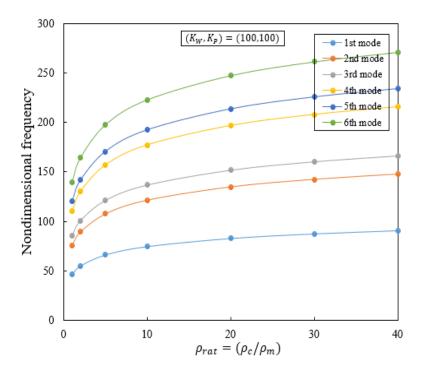
**Fig. 5.10 (d):** Effect of  $E_{rat}$  on frequency parameter ( $\omega^*$ ) for square E-FGM plate for constant  $(K_W, K_p = 100,100)$ ,  $\rho_{rat} = 2$ , and h/a = 0.01.



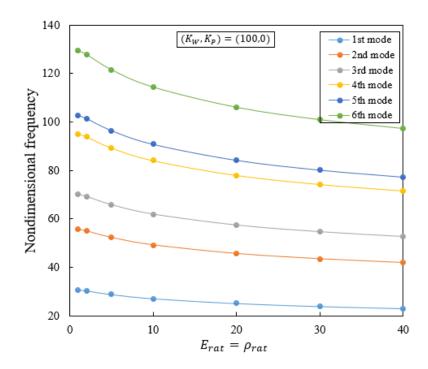
**Fig. 5.11 (a):** Effect of  $\rho_{rat}$  on frequency parameter ( $\omega^*$ ) for square E-FGM plate for constant  $(K_W, K_p = 100,0)$ ,  $E_{rat} = 2$ , and h/a = 0.01.



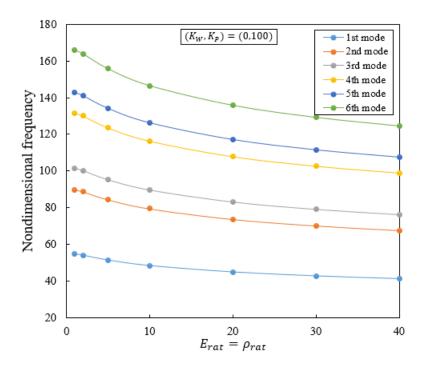
**Fig. 5.11 (b):** Effect of  $\rho_{rat}$  on frequency parameter ( $\omega^*$ ) for square E-FGM plate for constant  $(K_W, K_p = 0.100)$ ,  $E_{rat} = 2$  and h/a = 0.01.



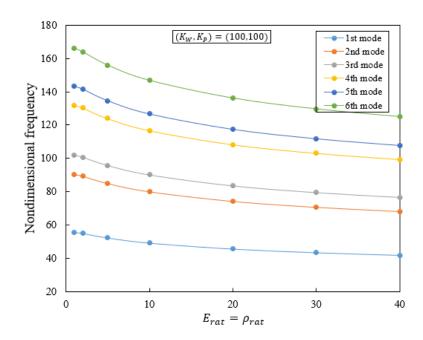
**Fig. 5.11 (c):** Effect of  $\rho_{rat}$  on frequency parameter ( $\omega^*$ ) for square E-FGM plate for constant  $(K_W, K_p = 100,100)$ ,  $E_{rat} = 2$  and h/a = 0.01.



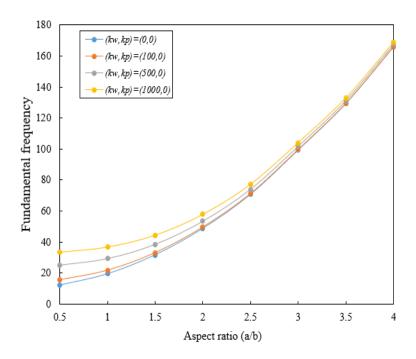
**Fig. 5.12** (a): Effect of  $E_{rat}$  and  $\rho_{rat}$  on frequency parameter ( $\omega^*$ ) for square E-FGM plate for constant  $(K_W, K_p = 100,0)$ ,  $(E_{rat} = \rho_{rat})$  and h/a = 0.01.



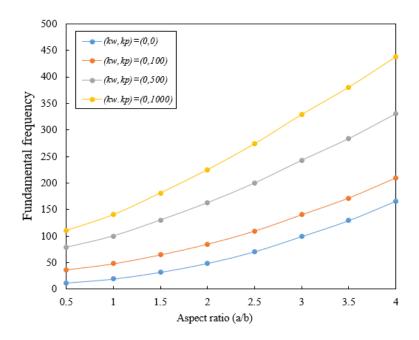
**Fig. 5.12** (b): Effect of  $E_{rat}$  and  $\rho_{rat}$  on frequency parameter ( $\omega^*$ ) for square E-FGM plate for constant  $(K_W, K_p = 0.100)$ ,  $(E_{rat} = \rho_{rat})$  and  $h/\alpha = 0.01$ .



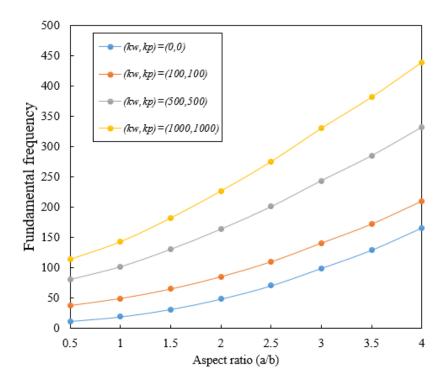
**Fig. 5.12** (c): Effect of  $E_{rat}$  and  $\rho_{rat}$  on frequency parameter ( $\omega^*$ ) for square E-FGM plate for constant  $(K_W, K_p = 100, 100)$ ,  $(E_{rat} = \rho_{rat})$  and h/a = 0.01.



**Fig. 5.13 (a):** Effect of aspect ratio on frequency parameter  $(\omega^*)$  under different combinations of  $K_W$  and fixed  $K_p=0$  for SSSS boundary conditions for  $E_{rat}=2$ ,  $\rho_{rat}=2$  and h/a=0.01.



**Fig. 5.13 (b):** Effect of aspect ratio on frequency parameter  $(\omega^*)$  under different combinations of  $K_p$  and fixed  $K_W = 0$  for SSSS boundary conditions for  $E_{rat} = 2$ ,  $\rho_{rat} = 2$  and h/a = 0.01.



**Fig. 5.13** (c): Effect of aspect ratio on natural frequency parameter ( $\omega^*$ ) under different combinations of  $K_W$  and  $K_p$  for SSSS boundary conditions for  $E_{rat}=2$ ,  $\rho_{rat}=2$ , and h/a=0.01.

#### **5.4. Summary**

In this chapter, the dynamic stiffness method is formulated for thin rectangular E-FGM plate resting on Winkler and Pasternak elastic foundation. The concept of a physical neutral surface is applied rather than the geometric middle surface with classical plate theory for the theoretical formulation of a thin E-FGM plate. The partial governing differential motion equation is obtained under the consideration of elastic foundation using Hamilton's principle. A Levy-type solution is applied with enforces displacement and force boundary conditions at the contrary edge of the plate. The Wittrick-Williams algorithm is implemented as a solution technique for solving the transcendental nature of the dynamic stiffness matrix to find out the natural frequencies of the E-FGM plate resting on Winkler and Pasternak elastic foundation.

The methodology of DSM has been imported into a MATLAB program. The natural frequency results obtained by DSM are compared with the published reported The present work contributes a net set of natural frequency results incorporating Winkler and Pasternak elastic foundation for square and rectangular E-FGM plates. It is also noticed that the impact of the Pasternak modulus on natural frequency is high than the Winkler modulus for various material property variations of the E-FGM plate. The effect of various material parameters, elastic foundation, and boundary conditions on natural frequency is also highlighted in the form of tables and graphs. The obtained natural frequency results by DSM resting on an elastic foundation can be further used as a benchmark solution for future comparison

# **CHAPTER 6**

### **Conclusions**

In this research work, the dynamic stiffness method (DSM) has been implemented along the application of the Wittrick-Williams algorithm to estimate the free vibration behavior of rectangular thin FGM plates with Winkler and Pasternak elastic foundation. This study considers three types of property variations (i.e., power-law, sigmoid law, and exponential law along the thickness direction of the FGM plates. In this work, the classical plate theory, in combination with the concept of the physical neutral surface, has been used to obtain the governing differential equation of motion of the FGM plate through the application of Hamilton's principle. A closed-form (Levy type) analytical solution is obtained from the governing differential equation of motion. With this approach, an exact dynamic stiffness matrix is formulated by applying force and displacement boundary conditions at appropriate edges of the FGM plate. The elements of this exact dynamic stiffness matrix are frequencydependent and explicit expressions of these elements are obtained for the different cases considered in the thesis. Finally, the well-established Wittrick-William algorithm is used to solve the assembled dynamic stiffness matrix of the FGM plate resting on the elastic foundation for very accurate computation of the plate's natural frequencies and mode shapes. The entire procedure of this dynamic stiffness formulation for different cases considered in this work is implemented in MATLAB to compute any number of natural frequencies and the associated mode shapes of the plate. Based on this study, some important conclusions pertaining to the free vibration characteristics of different cases involving different material property variations of FGM plates resting on the elastic foundation are highlighted below

#### 1. Analysis of P-FGM plate resting on elastic foundation:

■ This present work starts with the analysis of free vibration characterization of thin rectangular power-law (P-FGM) plates resting on the elastic foundation by using dynamic stiffness method (DSM). It is observed that the extract natural frequency results for P-FGM plates are in excellent agreement with published reported results on elastic foundations.

- Interestingly, it can be observed that the obtained natural frequency results for the P-FGM plate without elastic foundation for p=0 (i.e., isotropic plate) exactly match with the Leissa [265] reported results up to four decimal places value for all Levy-type (two opposite edges are simply-supported) boundary conditions. The reported literature validated the obtained natural frequency results for P-FGM rectangular plates resting on elastic foundations (i.e.,  $p \neq 0$ ). It is observed that the extract natural frequency results using DSM methodology are in excellent agreement with published literature results resting on elastic foundations.
- Under the study of comparative results, some inaccurate frequency results are reported in the existing literature [126]. The possible reason for these inaccurate results is also investigated. Instead of a geometrical mid surface, a physical neutral surface is used for the reference surface and model for the P-FGM plate in the existing literature, which leads to these inaccuracies. Due to this reason, the inaccuracy in the published frequency results becomes zero for a particular condition at p = 0 (i.e., isotropic homogeneous plate), and it consequently increases (higher) because of an increase in the values of p.
- The variation of free vibration frequencies with the change of parametric numerical values such as density ratio, modulus ratio, aspect ratio, volume fraction index, boundary conditions, and elastic foundation parameters on P-FGM plates resting on elastic foundations are also highlighted in the form of tables and graphs.
- As the aspect ratio increases, natural frequency increases with an increase in elastic foundation values. This increase in the frequency results is due to an increase in the flexural stiffness of the P-GFM plate due to an increase in the value of elastic foundations with more added metal constituents in the given structure.
- It is observed that the effect of p in volume fraction law (power-law) influences the plate's natural frequency but does not affect (or influence) the mode shapes of the plate. Hence, the P-FGM plate mode shapes do not change with the different values of p. considered the particular case of the isotropic homogeneous plate (i.e., p = 0). Thus for all p values, mode shapes of the P-FGM plate remain the same.
- It is noticed that the Pasternak modulus with different material and geometric parameters (material gradient index, aspect ratio, boundary condition) has more influence on the natural frequency than the Winkler modulus.

It can be highlighted further that the applied DSM processor for natural frequency estimation of P-FGM plate resting on elastic foundations is highly accurate. Therefore, obtained frequency results can be considered as a benchmarks solution for comparison purposes.

#### 2. Analysis of S-FGM resting on elastic foundation:

- In the subsequent investigation of the present thesis, the free vibration behavior of the sigmoid (S-FGM) functionally graded plate resting on elastic foundation has been reported by the dynamic stiffness method in conjunction with the application of the Wittrick-Williams algorithm. Here, the material property variation of the FGM plate is described by using the sigmoid law function. The natural frequencies of DSM methodology are compared and validated with the reported literature and found to be in excellent agreement.
- It is illustrated by validation with reported results that the implemented DSM has considerable advantages over the other reported analytical methods. Subsequently, DSM used the Wittrick and Williams algorithm to extract the natural frequency of the FGM plate and obtained no natural frequency is missed in the given plate under a particular frequency range.
- A new set of natural frequency results based on the proposed DSM methodology for thin square and rectangular S-FGM plates resting on elastic foundations are highlighted.
- It is noticed that the natural frequencies decrease with an increase in the sigmoid volume fraction indices (k) for a given Levy type's boundary condition and a given combination of elastic foundation. As k increases, stiffness of the S-FGM plate decrease, which means more metal, is introduced into the given structure.
- For Levy type boundary condition, it is also observed that the natural frequency increases with the aspect ratio and the elastic moduli of Winkler-Pasternak foundation for a given value of *k* for all six boundary conditions except for the SFSF boundary condition. This increase in natural frequency is because the stiffness of the plate increases due to changes in boundary conditions, geometry, and elastic foundations. But in the case of free (SFSF) boundary conditions, the plate becomes less stiff as the aspect ratio increases, due to which the natural frequency decreases.

- The natural frequency of a given mode increases with an increase in the elastic foundations. This is because of an increase in the stiffness of the plate structure, and it is also shown that the non-dimensional natural frequency is maximum for the boundary condition of SCSC and minimum for the boundary condition of SFSF. The reason is that the more constraint added at the boundary of the plate, the higher the stiffness of the plate, which causes an increase in the natural frequencies.
- The development of the S-FGM plate also helps to analyze the bimetallic nature of the plate. For the sigmoid law index, k = 0, the distribution of the material property of the S-FGM plate does not change in the transverse direction and is called a transversely isotropic homogeneous plate. When the value of the sigmoid law index (k) is very high, the FGM plate behaves as a bi-material in nature where the upper layer of the plate is highly rich in ceramic and the bottom layer is highly rich in metallic.

#### 3. Analysis of E-FGM resting on elastic foundation:

- The last part of the thesis presents the free vibration response of thin rectangular exponential-law (E-FGM) plates resting on elastic foundation using the dynamic stiffness method (DSM). It has been observed from the reported literature that very limited work has been reported on the free vibration behavior of E-FGM plate with DSM methodology. Therefore, in this present thesis, DSM has been used to solve the free vibration behavior of an E-FGM plate resting on elastic foundation.
- Some inaccurate natural frequencies reported results [126] for the E-FGM plates are highlighted. The possible reason for this inaccuracy is highlighted. Firstly, the author [126] implemented the Classical plate theory that considered the middle surface instead of the physical neutral surface as the reference plane of the E-FGM plate. Secondly, Ref. [126] applied the Rayleigh-Ritz method for their investigation. The accuracy of extracted natural frequency, particularly for a higher range (modes), is conceded if a sufficient number of trail functions are not taken, whereas DSM does not have such drawbacks.
- It is noticed that the E-FGM plate natural frequency results by the DSM are in excellent agreement compared with existing reported results. It is found that at  $\delta = 0$  plate is considered isotropic, and the obtained results match up to four decimal of the reported results by Leissa [265].

- The E-FGM plate's variations of free vibration frequencies with the change of parametric numerical values (density ratio, modulus ratio, aspect ratio, boundary conditions, and elastic foundation parameters) are reported.
- It is obtained from the frequency results of rectangular thin E-FGM plate that the Pasternak-Winkler elastic foundation increases the natural frequencies because of increasing the plate stiffness. It is noticed from the results that the effect of the Pasternak modulus is higher than the Winkler modulus on natural frequencies for all Levy-type boundary conditions.

The present thesis work can be summarized as the development of the dynamic stiffness method for analyzing the free vibration response of an FGM plate resting on elastic foundation. Here, DSM, along with classical plate theory and Wittrick-Williams algorithm, carried out exact natural frequency and mode shapes of the FGM plate. All three types of volume fraction laws are implemented in the analysis for the variation of material property in the FGM plate. All the possible parametric case study, along with the elastic foundation, is investigated for free vibration response of the FGM plate. Therefore, this present research thesis based on the free vibration response of an FGM plate resting on the elastic foundation can be measured as a valuable addition to the research field.

## **CHAPTER 7**

## Scope for the future research

The present research work analyzes the free vibration response of an FGM plate resting on elastic foundation. The dynamic stiffness method considers all three possible variations of material property. The kinematic variable of the FGM plate is developed using the classical plate theory. Therefore, the influence of shear deformation is not implemented, and the obtained results of the FGM plate generally may not be considered or applicable for thicker FGM plates. The Levy-type (displacement and force) boundary condition is implemented where two opposite sides of the plate are simply supported, and the other two sides are arbitrary conditions. Thus, this present research in the thesis has the potential ways to recommend future work. The formulation of the proposed DSM for FGM plates can be extended to cover the following topics.

- The DS matrix may be formulated to evaluate the response of free vibration behavior of FGM plate resting on elastic foundation using shear deformation theory.
- Formulation of dynamic stiffness method for free vibration behavior of cracked FGM plate resting on elastic foundation.
- Development of dynamic stiffness method for vibration response of sandwiches and circular FGM plates resting on elastic foundation.
- The investigation of free vibration response of FGM plate resting on elastic foundation considering thermal environment conditions with higher-order shear deformation theory by dynamic stiffness method.

### LIST OF PUBLICATIONS

### **International Journals**

- [1]. Manish Chauhan, Sarvagya Dwivedi, Ratneshwar Jha, Prabhakar Sathujoda, Vinayak Ranjan, (2021), "Sigmoid functionally graded plates embedded on Winkler-Pasternak foundation: free vibration analysis by dynamic stiffness method", Composite structures, Elsevier- SCI, (IF: 5.407), <a href="https://doi.org/10.1016/jcompstruct.2022">https://doi.org/10.1016/jcompstruct.2022</a>.1154 00
- [2]. Manish Chauhan, Pawan Mishra, Vinayak Ranjan,(2022) "Development of the Dynamic Stiffness Method for the Out-of-Plane Natural Vibration of an Orthotropic Plate" Applied Science, SCI, (IF): 2.769, https://doi.org/10.3390/app12115733
- [3]. Manish Chauhan, Pawan Mishra, Vinayak Ranjan,(2022) "Dynamic stiffness development for natural vibration analysis of thin power-law functionally graded plates supported on Winkler-Pasternak foundation" International Journal of Mechanical Science, Elsevier- SCI, Impact Factor: 5.329 (Under Review)
- [4]. Manish Chauhan, Pawan Mishra, Vinayak Ranjan, (2022) "Exponential functionally graded plates resting on Winkler-Pasternak foundation: free vibration analysis by dynamic stiffness method", Archive of Applied Mechanics, Springer- SCI, Impact Factor: 1.976 (Under Review)
- [5]. Manish Chauhan, Pawan Mishra, Vinayak Ranjan, (2022) "Out of plane free vibration analysis of rectangular functionally graded plate using dynamic stiffness method with Wittrick-William algorithm" Journal of Vibration Engineering and Technologies, Springer- SCI, Impact Factor: 1.889. (Under Review)

### **International Conferences**

- [1]. Manish Chauhan, Vinayak Ranjan (2021), paper presented on "Application of dynamic stiffness method and Wittrick-Williams algorithm for free vibration response of functionally graded plate",13-14, Oct, 2021, International online symposium on Aeroelasticity, Fluid-Structure Interaction, and Vibrations.
- [2]. Manish Chauhan, Vinayak Ranjan, Prabhakar Sathujoda, (2019), paper presented on "Dynamic stiffness method for free vibration analysis of thin functionally graded rectangular plates", 28-30, Nov, Bennett University and Published in Vibroengineering PROCEDIA, Vol.29, 2019, p. 76-81. https://doi.org/10.21595/vp.2019.21111.
- [3]. Manish Chauhan, Vinayak Ranjan, Baij Nath Singh, (2018), paper presented on "Comparison of Natural frequencies of isotropic plate using DSM with Wittrick-Williams Alogorithm',13-15, Dec 2018, Bennett University and Published in Vibroengineering PROCEDIA, Vol.21,2018, p.58-64. https://doi.org/10.21595/vp.2018.20401

# Appendix A

Case 2. 
$$\left(\alpha_m^2 - \frac{k_P \omega^2}{2D_{SFGM}}\right) < \sqrt{\left(\frac{k_P \omega^2}{2D_{SFGM}}\right)^2 + \frac{k_W \omega^2}{D_{SFGM}}}$$

For case 2, two roots  $(r_{1m}, -r_{1m})$  are real and two roots  $(ir_{2m}, -ir_{2m})$  are imaginary and are given by

$$r_{1m} = \sqrt{\left(\alpha_m^2 - \frac{k_P \omega^2}{2D_{SFGM}}\right) + \sqrt{\left(\frac{k_P \omega^2}{2D_{SFGM}}\right)^2 + \frac{k_W \omega^2}{D_{SFGM}}}}$$

$$r_{2m} = \sqrt{-\left(\alpha_m^2 - \frac{k_P \omega^2}{2D_{SFGM}}\right) + \sqrt{\left(\frac{k_P \omega^2}{2D_{SFGM}}\right)^2 + \frac{k_W \omega^2}{D_{SFGM}}}},$$
(A.1)

The solution is:

$$W_m(x) = A_m \cosh(r_{1m}x) + B_m \sinh(r_{1m}x) + C_m \cos(r_{2m}x) + D_m \sin(r_{2m}x)$$
(A.2)

Now, the displacement, w, is known, using the Eqs. (3.16) and (A.2), the bending rotation  $\phi_y$ , shear force  $V_x$  and moment  $M_{xx}$  can be extracted and are given by Eq. (3.12).

$$\phi_{y_m}(x,y) = \Phi_{y_m}(x)\sin(\alpha_m y)$$

$$= -\left( (A_m r_{1_m} \sinh(r_{1_m} x) + B_m r_{1_m} \cosh(r_{1_m} x) + C_m r_{2_m} \sin(r_{2_m} x) + D_m r_{2_m} \cos(r_{2_m} x) \right) \sin(\alpha_m y)$$
(A.3)

$$V_{x_m}(x,y) = V_{x_m}(x)\sin(\alpha_m y)$$

$$= -D_{SFGM}(A_m(r_{1m}^3 - (2 - \nu)\alpha_m^2 r_{1_m})\sinh(r_{1m}x) + B_m(r_{1m}^3 - (2 - \nu)\alpha_m^2 r_{1m})\cosh(r_{1m}x) + C_m(r_{2m}^3 + (2 - \nu)\alpha_m^2 r_{2m})\sin(r_{2m}x) + D_m(r_{2m}^3 + (2 - \nu)\alpha_m^2 r_{2m})\cos(r_{2m}x))\sin(\alpha_m y)$$
(A.4)

$$\begin{split} M_{xx_m}(x,y) &= \mathcal{M}_{xx_m}(x) \sin{(\alpha_m y)} \\ &= -D_{SFGM} \left( A_m \left( r_{1_m}^2 - \nu \alpha_m^2 \right) \cosh{\left( r_{1_m} x \right)} + B_m \left( r_{1_m}^2 - \nu \alpha_m^2 \right) \sinh{\left( r_{1_m} x \right)} \right. \\ &+ \mathcal{C}_m \left( r_{2,m}^2 - \nu \alpha_m^2 \right) \cos(r_{2m} x) + D_m \left( r_{2_m}^2 - \nu \alpha_m^2 \right) \sin{\left( r_{2m} x \right)} \right) \sin{\left( \alpha_m y \right)} \end{split}$$

(A.5)

The boundary conditions of displacements are given by

$$x = 0$$
  $W_m = W_1;$   $\phi_{y_m} = \phi_{y_1}$  
$$x = b$$
  $W_m = W_2;$   $\phi_{y_m} = \phi_{y_2}$  (A.6)

The force boundary conditions are given by

$$x = 0$$
  $V_{xm} = -V_1;$   $M_{xx_m} = -M_1$   $X = b$   $V_{xm} = V_2;$   $M_{xx_m} = M_2$  (A.7)

The displacement boundary conditions shown in Eq. (A.6) are substituted into solution Eq. (A.2) and rotation Eq. (A.3) to get the following matrix.

$$\begin{bmatrix} W_1 \\ \Phi_{y_1} \\ W_2 \\ \Phi_{y_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -r_{1_{1m}} & 0 & -r_{2m} \\ C_{h_1} & S_{h_1} & C_2 & S_2 \\ -r_{1_m} S_{h_1} & -r_{1_m} C_{h_1} & r_{2_m} S_2 & -r_{2_m} C_2 \end{bmatrix} \begin{bmatrix} A_m \\ B_m \\ C_m \\ D_m \end{bmatrix}$$
 (A.8)

or,

$$\delta = AC,$$
 (A.9)

where

$$C_{h_i} = \cosh(r_i b), \quad S_{h_i} = \sinh(r_i b)$$
  
 $C_i = \cos(r_i b), \quad S_i = \sin(r_i b) \text{ with } i = 1,2.$  (A.10)

A similar procedure is to be followed for force boundary conditions, i.e., Eq. (A.7) is substituted into shear force Eq. (A.4) and moment Eq. (A.5) to get the following matrix.

$$\begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} 0 & R_1 & 0 & R_2 \\ L_1 & 0 & L_2 & 0 \\ -R_1 S_{h_1} & -R_1 C_{h_1} & R_2 S_2 & -R_2 C_2 \\ -L_1 C_{h_1} & -L_1 S_{h_1} & -L_2 C_2 & -L_2 S_2 \end{bmatrix} \begin{bmatrix} A_m \\ B_m \\ C_m \\ D_m \end{bmatrix}$$
 (A.11)

or,

$$F = RC$$

(A.12)

where

$$R_i = D_{SFGM}[r_{im}^3 - \alpha^2 r_{im}(2 - \nu)], L_i = D_{SFGM}(r_{im}^3 - \alpha^2 \nu)$$
(A.13)

with i = 1,2.

For eliminating the constant vector value of C, the following relationship can be formed as

$$\mathbf{F} = \mathbf{K}\boldsymbol{\delta} \tag{A.14}$$

where

$$K = RA^{-1} \tag{A.15}$$

In the above expression, K represents the 4×4 square symmetric dynamic stiffness (DS) matrix containing the independent terms  $(S_{vv}, S_{vm}, F_{vv}, F_{vm}, S_{mm}, S_{vn})$ . Therefore, the obtained matrix (K) of the plate element can be given by

$$[K] = \begin{bmatrix} S_{vv} & S_{vm} & F_{vv} & F_{vm} \\ S_{vm} & S_{mm} & -F_{vm} & F_{mm} \\ F_{vv} & -F_{vm} & S_{vv} & -S_{vm} \\ F_{vm} & F_{mm} & -S_{vm} & S_{mm} \end{bmatrix}$$
 (A.16)

The explicit mathematical expressions of the six independent terms of this DS matrix are given below.

$$S_{wv} = (r_{2m}R_1 + r_{1m}R_2)(r_{2m}C_{h1}S_2 + r_{1m}C_2S_{h1})/\Delta,$$

$$S_{vm} = -r_{2m}\{R_1(C_2^2 - C_2C_{h1} + S_2^2) - R_2S_2S_{h1}\}$$

$$-r_{1m}\left\{R_1S_2S_{h1} - R_2\left((C_2 - C_{h1})C_{h1} + S_{h1}^2\right)\right\}/\Delta,$$

$$S_{mm} = (L_1 - L_2)(r_{1m}C_{h1}S_2 - r_{2m}C_2S_{h1})/\Delta,$$

$$f_{wv} = (r_{2m}R_1 - r_{1m}R_2)(r_{2m}S_2 + r_{1m}S_{h1})/\Delta,$$

$$f_{vm} = (C_{h2} - C_{h1})(r_{2m}R_1 - r_{1m}R_2)/\Delta$$

$$f_{mm} = (L_1 - L_2)(r_{2m}S_{h1} - r_{1m}S_{h2})/\Delta,$$

$$(A.17)$$

where  $r_{im}$ ,  $S_{hi}$ ,  $C_{hi}$ ,  $S_i$ ,  $C_i$ ,  $L_i$ ,  $R_i$  (with i = 1,2) are defined in Eqs. (A.1), (A.10) and (A.13), respectively and the other parameters can be expressed as:

$$R_{i} = D_{SFGM}[r_{im}^{3} - \alpha^{2}r_{im}(2 - v)] + r_{im}I_{2}\omega^{2},$$

$$L_{i} = D_{SFGM}(r_{im}^{3} - \alpha^{2}v) \text{ with } i = 1,2$$
(A.18)

and  $\Delta$  given by

$$\Delta = S_2 S_{h1} (r_{1m}^2 - r_{2m}^2) + r_{1m} r_{2m} \{ (C_2 - C_{h1})^2 + S_2^2 - S_{h1}^2 \}$$
(A.19)

# **APPENDIX B**

Case 2. 
$$\left(\alpha_m^2 - \frac{k_P \omega^2}{2D_{SFGM}}\right) < \sqrt{\left(\frac{k_P \omega^2}{2D_{SFGM}}\right)^2 + \frac{k_W \omega^2}{D_{SFGM}}}$$

For case 2, two roots  $(r_{1m}, -r_{1m})$  are real and two roots  $(ir_{2m}, -ir_{2m})$  are imaginary and are given by

$$r_{1m} = \sqrt{\left(\alpha_m^2 - \frac{k_P \omega^2}{2D_{SFGM}}\right) + \sqrt{\left(\frac{k_P \omega^2}{2D_{SFGM}}\right)^2 + \frac{k_W \omega^2}{D_{SFGM}}}}$$

$$r_{2m} = \sqrt{-\left(\alpha_m^2 - \frac{k_P \omega^2}{2D_{SFGM}}\right) + \sqrt{\left(\frac{k_P \omega^2}{2D_{SFGM}}\right)^2 + \frac{k_W \omega^2}{D_{SFGM}}}},$$
(B.1)

The solution is:

$$W_m(x) = A_m \cosh(r_{1m}x) + B_m \sinh(r_{1m}x) + C_m \cos(r_{2m}x) + D_m \sin(r_{2m}x)$$
 (B.2)

Now, the displacement, w, is known, using the Eqs. (4.16) and (B.2), the bending rotation  $\phi_y$ , shear force  $V_x$  and moment  $M_{xx}$  can be extracted and are given by Eq. (4.12).

$$\phi_{y_m}(x,y) = \Phi_{y_m}(x)\sin(\alpha_m y)$$

$$= -\left( (A_m r_{1_m} \sinh(r_{1_m} x) + B_m r_{1_m} \cosh(r_{1_m} x) + C_m r_{2_m} \sin(r_{2_m} x) + D_m r_{2_m} \cos(r_{2_m} x) \right) \sin(\alpha_m y)$$
(B.3)

$$V_{x_m}(x,y) = V_{x_m}(x)\sin(\alpha_m y)$$

$$= -D_{SFGM}(A_m(r_{1m}^3 - (2 - \nu)\alpha_m^2 r_{1_m})\sinh(r_{1m}x) + B_m(r_{1m}^3 - (2 - \nu)\alpha_m^2 r_{1m})\cosh(r_{1m}x) + C_m(r_{2m}^3 + (2 - \nu)\alpha_m^2 r_{2m})\sin(r_{2m}x) + D_m(r_{2m}^3 + (2 - \nu)\alpha_m^2 r_{2m})\cos(r_{2m}x))\sin(\alpha_m y)$$
(B.4)

$$\begin{split} M_{xx_m}(x,y) &= \mathcal{M}_{xx_m}(x)\sin\left(\alpha_m y\right) \\ &= -D_{SFGM} \Big( A_m \Big( r_{1_m}^2 - \nu \alpha_m^2 \Big) \cosh\left(r_{1_m} x\right) + B_m \Big( r_{1_m}^2 - \nu \alpha_m^2 \Big) \sinh\left(r_{1_m} x\right) \\ &+ \mathcal{C}_m \Big( r_{2,m}^2 - \nu \alpha_m^2 \Big) \cos(r_{2m} x) + D_m \Big( r_{2_m}^2 - \nu \alpha_m^2 \Big) \sin\left(r_{2m} x\right) \Big) \sin\left(\alpha_m y\right) \end{split}$$

(B.5)

The boundary conditions of displacements are given by

$$x = 0$$
  $W_m = W_1;$   $\phi_{y_m} = \phi_{y_1}$  
$$x = b$$
  $W_m = W_2;$   $\phi_{y_m} = \phi_{y_2}$  (B.6)

The force boundary conditions are given by

$$x = 0$$
  $V_{xm} = -V_1;$   $M_{xx_m} = -M_1$   $X = b$   $V_{xm} = V_2;$   $M_{xx_m} = M_2$  (B.7)

The displacement boundary conditions shown in Eq. (B.6) are substituted into solution Eq. (B.2) and rotation Eq. (B.3) to get the following matrix.

$$\begin{bmatrix} W_1 \\ \Phi_{y_1} \\ W_2 \\ \Phi_{y_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -r_{1_{1m}} & 0 & -r_{2m} \\ C_{h_1} & S_{h_1} & C_2 & S_2 \\ -r_{1_m} S_{h_1} & -r_{1_m} C_{h_1} & r_{2_m} S_2 & -r_{2_m} C_2 \end{bmatrix} \begin{bmatrix} A_m \\ B_m \\ C_m \\ D_m \end{bmatrix}$$
 (B.8)

or,

$$\delta = AC,$$
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where

$$C_{h_i} = \cosh(r_i b), \quad S_{h_i} = \sinh(r_i b)$$
  
 $C_i = \cos(r_i b), \quad S_i = \sin(r_i b) \text{ with } i = 1,2.$  (B.10)

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$$\begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} 0 & R_1 & 0 & R_2 \\ L_1 & 0 & L_2 & 0 \\ -R_1 S_{h_1} & -R_1 C_{h_1} & R_2 S_2 & -R_2 C_2 \\ -L_1 C_{h_1} & -L_1 S_{h_1} & -L_2 C_2 & -L_2 S_2 \end{bmatrix} \begin{bmatrix} A_m \\ B_m \\ C_m \\ D_m \end{bmatrix}$$
 (B.11)

or,

$$F = RC$$

(B.12)

where

$$R_i = D_{SFGM}[r_{im}^3 - \alpha^2 r_{im}(2 - \nu)], L_i = D_{SFGM}(r_{im}^3 - \alpha^2 \nu)$$
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with i = 1,2.

For eliminating the constant vector value of C, the following relationship can be formed as

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where

$$K = RA^{-1} \tag{A.15}$$

In the above expression, K represents the 4×4 square symmetric dynamic stiffness (DS) matrix containing the independent terms  $(S_{vv}, S_{vm}, F_{vv}, F_{vm}, S_{mm}, S_{vn})$ . Therefore, the obtained matrix (K) of the plate element can be given by

$$[K] = \begin{bmatrix} S_{vv} & S_{vm} & F_{vv} & F_{vm} \\ S_{vm} & S_{mm} & -F_{vm} & F_{mm} \\ F_{vv} & -F_{vm} & S_{vv} & -S_{vm} \\ F_{vm} & F_{mm} & -S_{vm} & S_{mm} \end{bmatrix}$$
 (B.16)

The explicit mathematical expressions of the six independent terms of this DS matrix are given below.

$$S_{wv} = (r_{2m}R_1 + r_{1m}R_2)(r_{2m}C_{h1}S_2 + r_{1m}C_2S_{h1})/\Delta,$$

$$S_{vm} = -r_{2m}\{R_1(C_2^2 - C_2C_{h1} + S_2^2) - R_2S_2S_{h1}\}$$

$$-r_{1m}\left\{R_1S_2S_{h1} - R_2\left((C_2 - C_{h1})C_{h1} + S_{h1}^2\right)\right\}/\Delta,$$

$$S_{mm} = (L_1 - L_2)(r_{1m}C_{h1}S_2 - r_{2m}C_2S_{h1})/\Delta,$$

$$f_{wV} = (r_{2m}R_1 - r_{1m}R_2)(r_{2m}S_2 + r_{1m}S_{h1})/\Delta,$$

$$f_{vm} = (C_{h2} - C_{h1})(r_{2m}R_1 - r_{1m}R_2)/\Delta$$

$$f_{mm} = (L_1 - L_2)(r_{2m}S_{h1} - r_{1m}S_{h2})/\Delta,$$
(B.17)

where  $r_{im}$ ,  $S_{hi}$ ,  $C_{hi}$ ,  $S_i$ ,  $C_i$ ,  $L_i$ ,  $R_i$  (with i = 1,2) are defined in Eqs. (B.1), (B.10) and (B.13), respectively and the other parameters can be expressed as:

$$R_{i} = D_{SFGM}[r_{im}^{3} - \alpha^{2}r_{im}(2 - v)] + r_{im}I_{2}\omega^{2},$$

$$L_{i} = D_{SFGM}(r_{im}^{3} - \alpha^{2}v) \text{ with } i = 1,2$$
(B.18)

and  $\Delta$  given by

$$\Delta = S_2 S_{h1} (r_{1m}^2 - r_{2m}^2) + r_{1m} r_{2m} \{ (C_2 - C_{h1})^2 + S_2^2 - S_{h1}^2 \}$$
(B.19)

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